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ELEMENTS OF GEOMETRY

PART II

HAMBLIN SMITH

1/6

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ELEMENTS OF GEOMET

PART II

CONTAINING THE

THIRD AND FOURTH BOOKS OF EUC

WITH

EXERCISES AND NOTES

BY

J. HAMBLIN SMITH, M.A

GONVILLE AND CAIUS COLLEGE, AND LATE LECTURER AT ST PETER'S COLL

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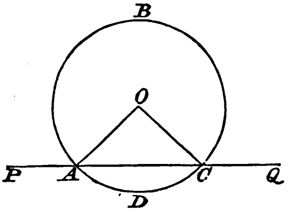
BOOK III.

POSTULATE.

A POINT is within, or without, a circle, according as its distance from the centre is less, or greater than, the radius of the circle.

DEFINITIONS.

I. A straight line, as *PQ*, drawn so as to cut a circle *ABCD*, is called a SECANT.

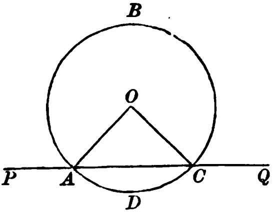


That such a line can only meet the circumference in *two* points may be shewn thus:

Some point within the circle is the centre; let this be *O*. Join *OA*. Then (Ex. 1, I. 16) we can draw one, and only one, straight line from *O*, to meet the straight line PQ, such that it shall be equal to *OA*. Let this line be OC. Then A and *C are the* only points in *PQ*, which are on the circumference *of the circle.*

II. The portion AC of the secant PQ, intercepted by the circle, is called a CHORD.

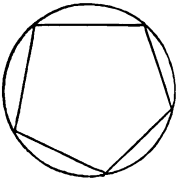
III. The two portions, into which a chord divides the circumference, as ABC and ADC, are called ARCS.



IV. The two figures into which a chord divides the circle, as ABC and ADC, that is, the figures, of which the boundaries are respectively the arc ABC and the chord AC, and the arc ADC and the chord AC, are called SEGMENTS of the circle.

V. The figure $AOCD$, whose boundaries are two radii and the arc intercepted by them, is called a SECTOR.

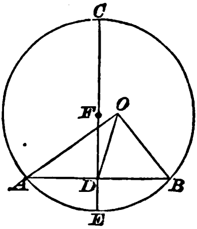
VI. A circle is said to be *described about* a rectilinear figure; when the circumference passes through



each of the angular points of the figure; and the figure is *said to be inscribed* in the circle.

PROPOSITION I. THEOREM.

The line, which bisects a chord of a circle at right angles, must contain the centre.



Let ABC be the given ⊙.

Let the st. line CE bisect the chord AB at rt. angles in D.

Then the centre of the ⊙ must lie in CE.

For if not, let O, a pt. out of CE, be the centre, and join OA, OD, OB.

Then, in △s ODA, ODB,

∵ $AD=BD$, and DO is common, and $OA=OB$;

∴ $\angle ODA = \angle ODB$; I. C.

∴ $\angle ODB$ is a right ∠. I. Def. 9.

But $\angle CDB$ is a right ∠, by construction;

∴ $\angle ODB = \angle CDB$, which is impossible;

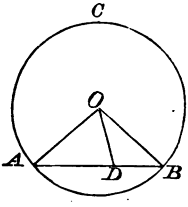
∴ O is not the centre.

Thus it may be shewn that no point, out of CE, can be the centre; and ∴ the centre must lie in CE.

Cor. If CE be bisected in F, then F is the centre *the circle.*

PROPOSITION II. THEOREM.

If any two points be taken in the circumference of a circle, the straight line, which joins them, must fall within the circle.



Let A and B be any two pts. in the ○ce of the ⊙ ABC.

Then must the st. line AB fall within the ⊙.

Take any pt. D in the line AB.

Find O the centre of the ⊙. III. 1, Cor.

Join OA, OD, OB.

Then $\because \angle OAB = \angle OBA$, I. 4.

and $\angle ODB$ is greater than $\angle OAB$, I. 16.

$\therefore \angle ODB$ is greater than $\angle OBA$;

and $\therefore OB$ is greater than OD. I. 19.

$\therefore$ the distance of D from O is less than the radius of the ⊙,

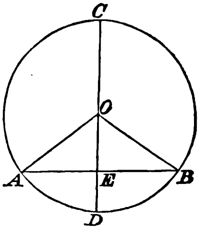
and $\therefore D$ lies within the ⊙.

And the same may be shewn of any other pt. in AB.

$\therefore AB$ lies entirely within the ⊙. Q. E. D.

PROPOSITION III. THEOREM.

If a straight line, drawn through the centre of a circle, bisect a chord of the circle, which does not pass through the centre, it must cut it at right angles: and conversely, if it cut it at right angles, it must bisect it.



In the ⊙ ABC, let the chord AB, which does not pass through the centre O, be bisected in E by the diameter CD.

Then must CD *be* ⊥ *to* AB.

Join OA, OB.

Then in △s AEO, BEO,

$\because AE = BE$, and EO is common, and $OA = OB$,

$\therefore \angle OEA = \angle OEB$. I. C.

Hence OE is ⊥ to AB, I. Def. 9.

that is, CD is ⊥ to AB.

Next, let CD be ⊥ to AB.

Then must CD *bisect* AB.

For $\because OA = OB$, and OE is common,

in the right-angled △s AEO, BEO,

$\therefore AE = BE$, I. E. Cor. p. 43.

that is, CD bisects AB. Q. E. D.

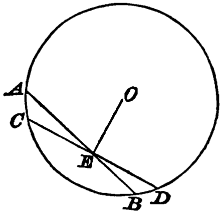
Ex. 1. Shew that if CD does not cut AB at right angles, it cannot bisect it.

Ex. 2. A line, which bisects two parallel chords in a circle, is also perpendicular to them.

Ex. 3. Through a given point within a circle, which is not the centre, draw a chord which shall be bisected in that point.

PROPOSITION IV. THEOREM.

If in a circle two chords, which do not both pass through the centre, cut one another, they do not bisect each other.



Let the chords AB, CD, which do not both pass through the centre, cut one another, in the pt. E, in the ⊙ $ACBD$.

Then AB, CD *do not bisect each other.*

If one of them pass through the centre, it is plainly not bisected by the other, which does not pass through the centre.

But if neither pass through the centre, find the centre O, and join OE.

Then ∵ OE, passing through the centre, bisects AB,

∴ $\angle OEA$ is a rt. $\angle$. III. 3.

And ∵ OE, passing through the centre, bisects CD,

∴ $\angle OEC$ is a rt. $\angle$; III. 3

∴ $\angle OEA = \angle OEC$, which is impossible;

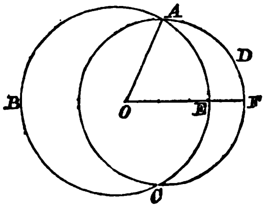
∴ AB, CD do not bisect each other. Q. E. D.

Ex. 1. Shew that the locus of the points of bisection of all parallel chords of a circle is a straight line.

Ex. 2. Shew that no parallelogram, except those which are *rectangular,* can be inscribed in a circle.

PROPOSITION V. THEOREM.

If two circles cut one another, they cannot have the same centre.



If it be possible, let O be the common centre of the ⊙s ABC, ADC, which cut one another in the pts. A and C.

Join OA, and draw OEF meeting the ⊙s in E and F.

Then ∵ O is the centre of ⊙ ABC,

∴ $OE = OA$;

and ∵ O is the centre of ⊙ ADC,

∴ $OF = OA$;

∴ $OE = OF$, which is impossible;

∴ O is not the common centre. Q. E. D.

Ex. Two circles, whose centres are A and B, intersect in in C; through C two chords DCE, FCG are drawn equally inclined to AB and terminated by the circles: prove that DE and FG are equal.

Note. Circles which have the same centre are called *Concentric.*

Note I. *On the contact of circles.*

DEF. VII. *Circles are said to touch each other, which meet but do not cut each other.*

One circle is said to touch another *internally*, when one point of the circumference of the former lies *on*, and no point *without*, the circumference of the other.

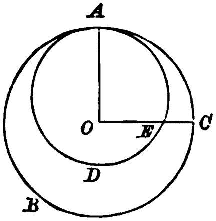
Hence for internal contact one circle must be smaller than the other.

Two circles are said to touch *externally*, when one point of the circumference of the one lies *on*, and no point *within* the circumference of the other.

No restriction is placed by these definitions on the number of points of contact, and it is not till we reach Prop. xiii that we prove that there can be *but one point of contact.*

PROPOSITION VI. THEOREM.

If one circle touch another internally, they cannot have the same centre.



Let ⊙ *ADE* touch ⊙ *ABC* internally,

and let *A* be a point of contact.

Then *some* point *E* in the ○ce *ADE* lies *within* ⊙ *ABC*. Def. 7.

If it be possible, let *O* be the common centre of the two ⊙s.

Join *OA*, and draw *OEC* meeting the ○ces in *E* and *C*.

Then ∵ *O* is the centre of ⊙ *ABC*,

∴ $OA = OC$; I. Def. 13.

and ∵ *O* is the centre of ⊙ *ADE*,

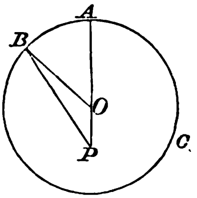
∴ $OA = OE$. I. Def. 13.

Hence $OE = OC$, which is impossible;

∴ *O* is not the common centre of the two ⊙s. Q. E. D.

PROPOSITION VII. THEOREM.

If from any point within a circle, which is not the centre straight lines be drawn to the circumference, the greatest of these lines is that which passes through the centre.



Let ABC be a ⊙, of which O is the centre.

From P, any pt. within the ⊙, draw PA, passing through O and meeting the ○ce in A.

Then must PA *be greater than any other line, drawn from* P *to the* ○*ce.*

For let PB be any other line, drawn from P to meet the ○ce in B, and join BO.

Then $\because AO = BO$,

$\therefore AP =$ sum of BO and OP.

But the sum of BO and OP is greater than BP, I. 20

and $\therefore AP$ is greater than BP. Q. E. D

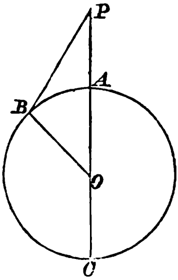
Ex. 1. If AP be produced to meet the circumference in D, shew that PD is less than any other line that can be drawn from P to the circumference.

Ex. 2. Shew that PB continually decreases, as B passes from A to D.

Ex. 3. Shew that two lines, but not three, that shall be equal, can be drawn from P to the circumference.

PROPOSITION VIII. THEOREM.

If from any point without a circle straight lines be drawn to the circumference, the least of these lines is that which, when produced, passes through the centre, and the greatest is that which passes through the centre.



Let ABC be a ⊙, of which O is the centre.

From P any pt. outside the ⊙, draw $PAOC$, meeting the ○ce in A and C.

Then must PA *be less, and* PC *greater, than any other line drawn from* P *to the* ○*ce.*

For let PB be any other line drawn from P to meet the ○ce in B, and join BO.

Then $\because$ sum of PB and BO is greater than OP, I. 20.

$\therefore$ sum of PB and BO is greater than sum of AP and AO.

But $BO = AO$;

$\therefore$ PB is greater than AP.

Again, $\because$ PB is less than the sum of PO, OB, I. 20.

$\therefore$ PB is less than the sum of PO, OC;

$\therefore$ PB is less than PC.

Ex. 1. Shew that PB continually increases as B passes from A to C.

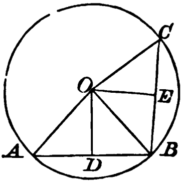
Ex. 2. Shew that from P two lines, but not three, that shall be equal, can be drawn to the circumference.

NOTE. From Props. VII. and VIII. we deduce the following Corollary, which we shall use in the proof of Props. XI. and XIII.

COR. *If a point be taken, within or without a circle, of all lines drawn from that point to* the circumference, the *greatest is that which meets the* circumference after passing *through the centre.*

PROPOSITION IX. THEOREM.

If a point be taken within a circle, from which there fall more than two equal straight lines to the circumference, that point is the centre of the circle.



Let O be a pt. in the ⊙ ABC, from which more than two st. lines OA, OB, OC, drawn to the ○ce, are equal.

Then must O *be the centre of the* ⊙.

Join AB, BC, and draw OD, OE ⊥ to AB, BC.

Then ∵ $OA = OB$, and OD is common,

in the right-angled △s AOD, BOD,

∴ $AD = DB$; I. E. Cor. p. 43.

∴ the centre of the ⊙ is in DO. III. 1.

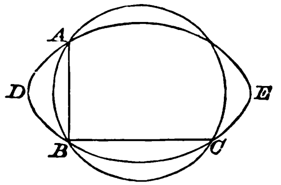
Similarly it may be shewn that

the centre of the ⊙ is in EO;

∴ O is the centre of the ⊙. Q. E. D.

PROPOSITION X. THEOREM.

Two circles cannot have more than two points common to both without coinciding entirely.



If it be possible, let *ABC* and *ADE* be two ⊙s which have more than two pts. in common, as *A*, *B*, *C*.

Join *AB*, *BC*.

Then ∵ *AB* is a chord of each circle,

∴ the centre of each circle lies in the straight line, which bisects *AB* at right angles; III. 1.

and ∵ *BC* is a chord of each circle,

∴ the centre of each circle lies in the straight line, which bisects *BC* at right angles. III. 1.

∴ the centre of each circle is the point, in which the two straight lines, which bisect *AB* and *BC* at right angles, meet.

∴ the ⊙s *ABC*, *ADE* have a common centre, which is impossible; III. 5 and 6.

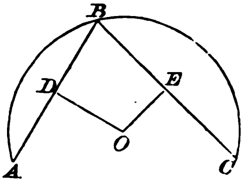
∴ two ⊙s cannot have more than two pts. common to both.

Q. E. D.

NOTE. We here insert two Propositions, Eucl. III. 25 and IV. 5, which are closely connected with Theorems I. and X. of this book. The learner should compare with this portion of the subject the note on Loci, Part I. p. 103.

PROPOSITION A. PROBLEM. (Eucl. III. 25.)

An arc of a circle being given, to complete the circle which it is a part.



Let ABC be the given arc.

It is required to complete the ⊙ *of which* ABC *is a par*

Take B, any pt. in arc ABC, and join AB, BC.

From D and E, the middle pts. of AB and BC,

draw DO, EO, ⊥s to AB, BC, meeting in O.

Then ∵ AB is to be a chord of the ⊙,

∴ centre of the ⊙ lies in DO; III. 1

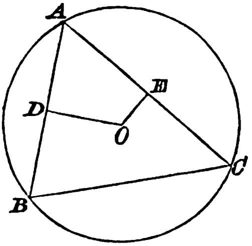
and ∵ BC is to be a chord of the ⊙,

∴ centre of the ⊙ lies in EO. III. 1

Hence O is the centre of the ⊙ of which ABC is an arc and if a ⊙ be described, with centre O and radius OA, thi will be the ⊙ required. Q. E. F

PROPOSITION B. PROBLEM. (Eucl. IV. 5.)

To describe a circle about a given triangle.



Let ABC be the given △.

It is required to describe a ⊙ about the △.

From D and E, the middle pts. of AB and BC, draw DO, EO, ⊥s to AB, BC, and let them meet in O.

Then ∵ AB is to be a chord of the ⊙,

∴ centre of the ⊙ lies in DO. III. 1.

And ∵ BC is to be a chord of the ⊙,

∴ centre of the ⊙ lies in EO. III. 1.

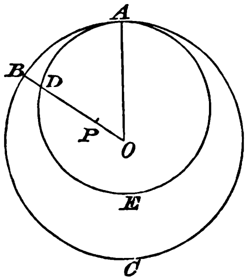
Hence O is the centre of the ⊙ which can be described about the △, and if a ⊙ be described with centre O and radius OA, this will be the ⊙ required. Q. E. F.

Ex. 1. If BAC be a right angle, shew that O will coincide with the middle point of BC.

Ex. 2. If BAC be an obtuse angle, shew that O will fall on the side of BC remote from A.

Proposition XI. Theorem.

If one circle touch another internally at any point, the centre of the interior circle must lie in that radius of the other circle which passes through that point of contact.



Let the ⊙ *ADE* touch the ⊙ *ABC* internally, and let *A* be a pt. of contact.

Find *O* the centre of ⊙ *ABC*, and join *OA*.

Then must the centre of ⊙ ADE *lie in the radius* OA.

For if not, let *P* be the centre of *ADE*.

Join *OP*, and produce it to meet the ○ces in *D* and *B*.

Then ∵ *P* is the centre of *ADE*, and from *O* are drawn to the ○ce of *ADE* the st. lines *OA*, *OD*, of which *OD* passes through *P*,

∴ *OD* is greater than *OA*. III. 8, Cor.

But $OA = OB$;

∴ *OD* is greater than *OB*,

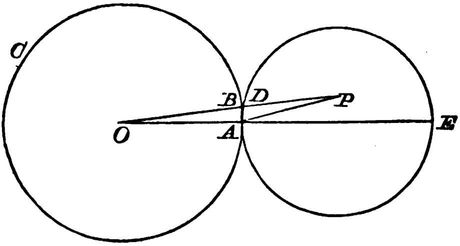
which is impossible.

∴ the centre of *ADE* is not out of the radius *OA*,

∴ it lies in *OA*. Q.E.D.

PROPOSITION XII. THEOREM.

If two circles touch one another externally at any point, the straight line joining the centre of one with that point of contact must when produced pass through the centre of the other.



Let ⊙ ABC touch ⊙ ADE externally in the pt. A.

Let O be the centre of ⊙ ABC.

Join OA, and produce it to E.

Then must the centre of ⊙ ADE *lie in* AE.

For if not, let P be the centre.

Join OP meeting the ⊙s in B, D; and join AP.

Then ∵ $OB=OA$,

and $PD=AP$,

∴ OB and PD together $=OA$ and AP together;

∴ OP is not less than OA and AP together.

But OP is less than OA and AP together, I. 20.

which is impossible;

∴ the centre of ⊙ ADE cannot lie out of AE. Q. E. D.

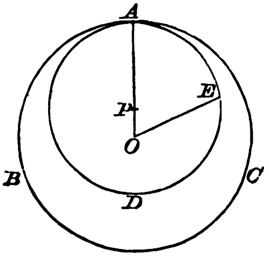
Ex. Three circles touch one another externally, whose centres are A, B, C. Shew that the difference between AB and AC is half as great as the difference between the diameters of the circles, whose centres are B and C.

Proposition XIII. Theorem.

One circle cannot touch another in more points than one, whether it touch it internally or externally.

First let the ⊙ ADE touch the ⊙ ABC *internally* in pt. A.

Then there can be no other point of contact.



Take O the centre of ABC.

Then P the centre of ADE lies in OA. III. 11.

Take any pt. E in the ○ce of the ⊙ ADE, and join OE.

Then ∵ from O, a pt. within or without the ⊙ ADE, two lines OA, OE are drawn to the ○ce, of which OA passes through the centre P,

∴ OA is greater than OE, III. 8, Cor.

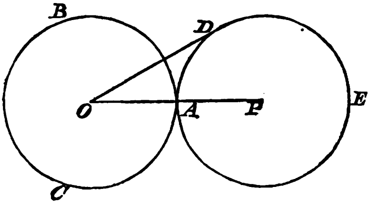
and ∴ E is a point *within* the ⊙ ABC.

Similarly it may be shewn that every pt. of the ○ce of ADE, except A, lies *within* the ⊙ ABC;

∴ A is the only point in which the ⊙s meet.

Next, let the ⊙s *ABC*, *ADE* touch *externally* in the pt. *A*.

Then there can be no other point of contact.



Take *O* the centre of the ⊙ *ABC*.

Then *P*, the centre of the ⊙ *ADE*, lies in *OA* produced. III. 12.

Take any pt. *D* in the ○ce of the ⊙ *ADE*, and join *OD*.

Then ∵ from *O*, a pt. outside the ⊙ *ADE*, two lines *OA*, *OD* are drawn to the ○ce, of which *OA* when produced passes through the centre *P*,

∴ *OD* is greater than *OA*; III. 8.

∴ *D* is a point *without* the ⊙ *ABC*.

Similarly, it may be shewn that every pt. of the ○ce of *ADE*, except *A*, lies *without* the ⊙ *ABC*;

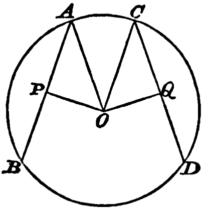
∴ *A* is the only point in which the ⊙s meet. Q. E. D.

DEF. VIII. The DISTANCE of a chord from the centre is measured by the length of the perpendicular drawn from the centre to the chord.

10—2

Proposition XIV. Theorem.

Equal chords in a circle are equally distant from centre; and conversely, those, which are equally dist from the centre, are equal to one another.



Let the chords AB, CD in $\odot$ $ABDC$ be equal.

Then must AB *and* CD *be equally distant from t centre* O.

Draw OP and OQ $\perp$ to AB and CD; and join AO, CO.

Then P and Q are the middle pts. of AB and CD: III.

and $\because AB = CD$, $\therefore AP = CQ$.

Then $\because AP = CQ$, and $AO = CO$,

in the right-angled $\triangle$s AOP, COQ,

$\therefore OP = OQ$; I. E. Cor. p. 4

and $\therefore$ AB and CD are equally distant from O. Def. VI

Next, let AB and CD be equally distant from O.

Then must AB = CD.

For $\because OP = OQ$, and $AO = CO$,

in the right-angled $\triangle$s AOP, COQ,

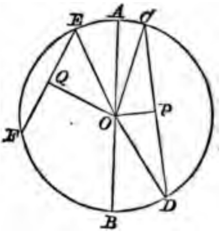
$\therefore AP = CQ$, I. E. Co

and $\therefore AB = CD$. Q. E.

Ex. In a circle, whose diameter is 10 inches, a chord drawn, which is 8 inches long. If another chord be draw at a distance of 3 inches from the centre, shew whether it equal or not to the former.

PROPOSITION XV. THEOREM.

The diameter is the greatest chord in a circle, and of all others that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.



Let AB be a diameter of the ⊙ $ABDC$, whose centre is O, and let CD be any other chord, not a diameter, in the ⊙, nearer to the centre than the chord EF.

Then must AB *be greater than* CD, *and* CD *greater than* EF.

Draw OP, OQ ⊥ to CD and EF; and join OC, OD, OE.

Then ∵ $AO = CO$, and $OB = OD$, I. Def. 13.

∴ AB = sum of CO and OD,

and ∴ AB is greater than CD. I. 20.

Again, ∵ CD is nearer to the centre than EF,

∴ OP is less than OQ. Def. VIII.

Now ∵ sq. on OC = sq. on OE,

∴ sum of sqq. on OP, PC = sum of sqq. on OQ, QE. I. 47.

But sq. on OP is less than sq. on OQ;

∴ sq. on PC is greater than sq. on QE;

∴ PC is greater than QE;

and ∴ CD is greater than EF.

Next, let CD be greater than EF.

Then must CD *be nearer to the centre than* EF.

For ∵ CD is greater than EF,

∴ PC is greater than QE.

Now sum of sqq. on OP, PC = sum of sqq. on OQ, QE.

But sq. on PC is greater than sq. on QE;

$\therefore$ sq. on OP is less than sq. on OQ;

$\therefore$ OP is less than OQ;

and $\therefore$ CD is nearer to the centre than EF. Q. E. D.

Ex. 1. Draw a chord of given length in a given circle, which shall be bisected by a given chord.

Ex. 2. If two isosceles triangles be of equal altitude, and the sides of one be equal to the sides of the other, shew that their bases must be equal.

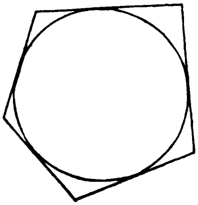
Ex. 3. Any two chords of a circle, which cut a diameter in the same point and at equal angles, are equal to one another.

DEF. IX. *A straight line is said to be a* TANGENT *to, or to touch, a circle, when it meets and, being produced, does not cut the circle.*

From this definition it follows that the tangent meets the circle in one point only, for if it met the circle in two points it would cut the circle, since the line joining two points in the circumference is, being produced, a secant. (III. 2).

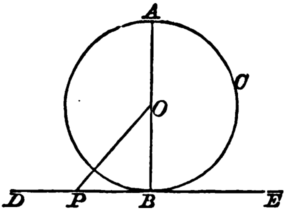
DEF. X. If from any point in a circle a line be drawn at right angles to the tangent at that point, the line is called a NORMAL to the circle at that point.

DEF. XI. A rectilinear figure is said to be *described about* a circle when each side of the figure touches the circle; and the circle is said to be *inscribed* in the figure.



PROPOSITION XVI. THEOREM.

The straight line drawn at right angles to the diameter of a circle, from the extremity of it, is a tangent to the circle.



Let ABC be a ⊙, of which the centre is O, and the diameter AOB.

Draw DE at right angles to AOB.

Then must DE *be a tangent to the* ⊙.

Take any point P in DE, and join OP.

Then, ∵ ∠ OBP is a right angle,

∴ ∠ OPB is less than a right angle, I. 17.

and ∴ OP is greater than OB. I. 19.

Hence P is a point outside the ⊙ ABC.

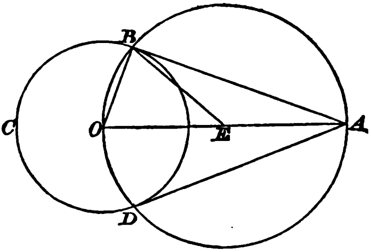
In the same way it may be shewn that every point in DE, or DE produced in either direction, except the point B, lies outside the ⊙;

∴ DE is a tangent to the ⊙. Def. IX.

Q. E. D.

PROPOSITION XVII. PROBLEM.

To draw a straight line from a given point, either WITHOUT *or* ON *the circumference, which shall touch a given circle.*



Let A be the given pt., *without* the ⊙ BCD.

Let O be the centre of ⊙ BCD, and join OA.

Bisect OA in E, and with centre E and radius EO, describe a ⊙ $ABOD$, cutting the given ⊙ in B and D.

Join AB, AD. *These are tangents to the* ⊙ BCD.

Join BO, BE.

Then $\because OE = BE$, $\therefore \angle OBE = \angle BOE$;

$\therefore \angle AEB =$ twice $\angle OBE$; I. 32.

and $\because AE = BE$, $\therefore \angle ABE = \angle BAE$;

$\therefore \angle OEB =$ twice $\angle ABE$; I. 32.

$\therefore$ sum of $\angle$s AEB, OEB = twice sum of $\angle$s OBE, ABE, that is, two right angles = twice $\angle OBA$;

$\therefore \angle OBA$ is a right angle,

and $\therefore AB$ is a tangent to the ⊙ BCD. I. 16.

Similarly it may be shewn that AD is a tangent to ⊙ BCD.

Next, let the given pt. be *on* the ○ce of the ⊙, as B.

Then, if BA be drawn ⊥ to the radius OB,

BA is a tangent to the ⊙ at B. I. 16.

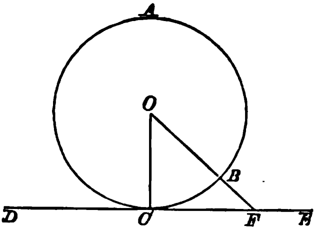
Q. E. D.

Ex. 1. Shew that the two tangents drawn from a point without the circumference to a circle are equal.

Ex. 2. If a quadrilateral $ABCD$ be described about a circle, shew that the sum of AB and CD is equal to the sum of AC and BD.

PROPOSITION XVIII. THEOREM.

If a straight line touch a circle, the straight line drawn from the centre to the point of contact must be perpendicular to the line touching the circle.



Let the st. line *DE* touch the ⊙ *ABC* in the pt. *C.*

Find *O* the centre, and join *OC.*

Then must OC *be* ⊥ *to* DE.

For if it be not, draw *OBF* ⊥ to *DE,* meeting the ○ce in *B.*

Then ∵ *OFC* is a rt. angle,

∴ *OCF* is less than a rt. angle, I. 17.

and ∴ *OC* is greater than *OF.* I. 19.

But *OC*=*OB,*

∴ *OB* is greater than *OF,* which is impossible;

∴ *OF* is not ⊥ to *DE,* and in the same way it may be shewn that no other line drawn from *O,* but *OC,* is ⊥ to *DE*;

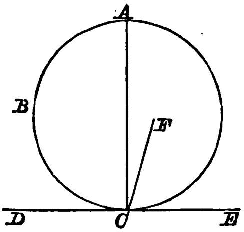
∴ *OC* is ⊥ to *DE.* Q. E. D.

Ex. If two straight lines intersect, the centres of all circles touched by both lines lie in two lines at right angles to each other.

NOTE. Prop. XVIII. might be stated thus:—*All radii of a circle are normals to the circle at the points where they meet the circumference.*

PROPOSITION XIX. THEOREM.

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle must be in that line.



Let the st. line *DE* touch the ⊙ *ABC* at the pt. *C*, and from *C* let *CA* be drawn ⊥ to *DE*.

Then must the centre of the ⊙ *be in* CA.

For if not, let *F* be the centre, and join *FC*.

Then ∵ *DCE* touches the ⊙, and *FC* is drawn from centre to pt. of contact,

∴ *FCE* is a rt. angle. III. 18.

But *ACE* is a rt. angle;

∴ ∠ *FCE* = ∠ *ACE*, which is impossible.

In the same way it may be shewn that no pt. out of *CA* can be the centre of the ⊙;

∴ the centre of the ⊙ lies in *CA*. Q. E. D.

Ex. Two concentric circles being described, if a chord of the greater touch the less, the parts intercepted between the two circles are equal.

Note. *Prop.* XIX. might be stated thus:—*Every normal to a circle passes through the centre.*

PROPOSITION XX. THEOREM.

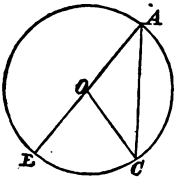
The angle at the centre of a circle is double of the angle at the circumference, subtended by the same arc.

Let ABC be a ⊙, O the centre,

BC any arc, A any pt. in the ○ce.

Then must $\angle\ BOC =$ *twice* $\angle\ BAC.$

First, suppose O to be in one of the lines containing the $\angle\ BAC$.



Then ∵ $OA = OC$,

∴ $\angle\ OCA = \angle\ OAC$; I. A.

∴ sum of $\angle$s OCA, OAC = twice $\angle\ OAC$.

But $\angle\ BOC$ = sum of $\angle$s OCA, OAC, I. 32.

∴ $\angle\ BOC$ = twice $\angle\ OAC$,

that is, $\angle\ BOC$ = twice $\angle\ BAC$.

Next, suppose O to be within (fig. 1), or without (fig. 2) the $\angle\ BAC$.

Fig. 1.

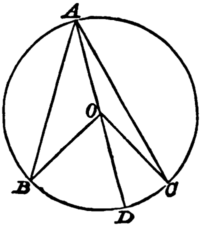
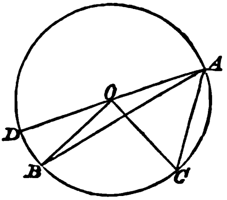


Fig. 2.



Join AO, and produce it to meet the ○ce in D.

Then, as in the first case,

$$\angle\ COD = \text{twice } \angle\ CAD,$$

$$\text{and } \angle\ BOD = \text{twice } \angle\ BAD;$$

∴, fig. 1, sum of $\angle$s COD, BOD = twice sum of $\angle$s CAD, BAD,

$$\text{that is, } \angle\ BOC = \text{twice } \angle\ BAC.$$

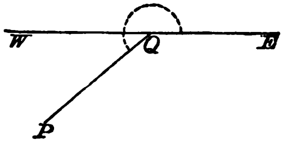
And, fig. 2, difference of $\angle$s COD, BOD = twice difference of $\angle$s CAD, BAD, that is, $\angle\ BOC$ = twice $\angle\ BAC$.

Q. E. D.

Ex. 1. The centre of the circle *CBED* is on the circumference of *ABD*. If from any point *A* the lines *ABC* and *AED* be drawn to cut the circles, the chord *BE* is parallel to *CD*.

Ex. 2. From any point in a straight line touching a circle a straight line is drawn through the centre, and is terminated by the circumference; the angle between these two straight lines is bisected by a straight line, which intersects the straight line joining their extremities. Shew that the angle between the last two lines is half a right angle.

Note II. We have already explained (Note iii. Book i. p. 28) how Euclid's definition of an angle may be extended with advantage, so as to include the conception of an angle equal to two right angles: and we now proceed to shew how the Definition given in that Note may be extended, so as to embrace angles greater than two right angles.



Let *WQ* be a straight line, and *QE* its continuation.

Then, by the Definition, the angle made by *WQ* and *QE*, which we propose to call a Flat Angle, is equal to two right angles.

Now suppose *QP* to be a straight line, which revolves about the fixed point *Q*, and which at first coincides with *QE*.

When *QP*, revolving from right to left, coincides with *QW*, it has described an angle equal to two right angles.

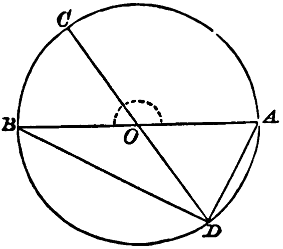
When *QP* has continued its revolution, so as to come into the position indicated in the diagram, it has described an angle *EQP*, indicated by the dotted line, greater than two right angles, and this we call a Reflex Angle.

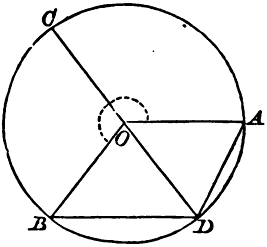
To assist the learner, we shall mark these angles with dotted lines in the diagrams.

Admitting the existence of angles, equal to and greater than two right angles, the Proposition last proved may be extended, as we now proceed to shew.

PROPOSITION C. THEOREM.

The angle, not less than two right angles, at the centre of a circle is double of the angle at the circumference, subtended by the same arc.





In the $\odot$ $ACBD$, let the angles AOB (not less than two right angles) at the centre, and ADB at the circumference, be subtended by the same arc ACB.

Then must $\angle AOB =$ *twice* $\angle ADB$.

Join DO, and produce it to meet the arc ACB in C.

Then $\because$ $\angle AOC =$ twice $\angle ADO$, III. 20.

and $\angle BOC =$ twice $\angle BDO$, III. 20.

$\therefore$ sum of $\angle$s AOC, BOC = twice sum of $\angle$s ADO, BDO,

that is, $\angle AOB =$ twice $\angle ADB$. Q. E. D.

NOTE. In fig. 1, AOB is drawn a flat angle,

and in fig. 2, AOB is drawn a reflex angle.

DEF. XII. The angle in a segment is the angle contained *by two straight lines* drawn from any point in the arc to the *extremities of the* chord.

PROPOSITION XXI. THEOREM.

The angles in the same segment of a circle are equal to one another.

Fig. 1.

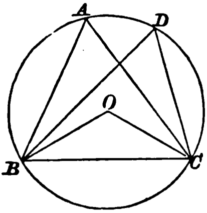
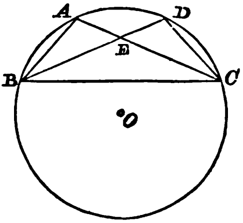


Fig. 2.



Let BAC, BDC be angles in the same segment $BADC$.

Then must $\angle\, BAC = \angle\, BDC$.

First, when segment $BADC$ is greater than a semicircle,

From O, the centre, draw OB, OC. (Fig. 1).
Then, $\because\ \angle\, BOC =$ twice $\angle\, BAC$, III. 20.
and $\angle\, BOC =$ twice $\angle\, BDC$, III. 20.
$\therefore\ \angle\, BAC = \angle\, BDC$.

Next, when segment $BADC$ is less than a semicircle,

Let E be the pt. of intersection of AC, DB. (Fig. 2).
Then, $\because\ \angle\, ABE = \angle\, DCE$, by the first case,
and $\angle\, BEA = \angle\, CED$, I. 15.
$\therefore\ \angle\, EAB = \angle\, EDC$, I. 32.
that is, $\angle\, BAC = \angle\, BDC$. Q. E. D.

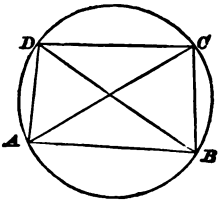
Ex. 1. Shew that, by assuming the possibility of an angle being greater than two right angles, both the cases of this proposition may be included in one.

Ex. 2. AB, AC are chords of a circle, D, E the middle points of their arcs. If DE be joined, shew that it will cut off equal parts from AB, AC.

Ex. 3. If two straight lines whose extremities are in the circumference of a circle cut one another, the triangles formed by joining their extremities are equiangular to each other.

PROPOSITION XXII. THEOREM.

The opposite angles of any quadrilateral figure, inscribed in a circle, are together equal to two right angles.



Let $ABCD$ be a quadrilateral fig. inscribed in a ⊙.

Then must each pair of its opposite ∠ *s be together equal to two rt.* ∠ *s.*

Draw the diagonals AC, BD.

Then ∵ $\angle ADB = \angle ACB$, in the same segment, III. 21.

and $\angle BDC = \angle BAC$, in the same segment; III. 21.

∴ sum of ∠s ADB, BDC = sum of ∠s ACB, BAC;

that is, $\angle ADC$ = sum of ∠s ACB, BAC.

Add to each $\angle ABC$.

Then ∠s ADC, ABC together = sum of ∠s ACB, BAC, ABC;

and ∴ ∠s ADC, ABC together = two right ∠s. I. 32.

Similarly, it may be shewn,

that ∠s BAD, BCD together = two right ∠s.

Q. E. D.

NOTE. Another method of proving this proposition is *given on page 209.*

Ex. 1. If one side of a quadrilateral figure inscribed in a circle be produced, the exterior angle is equal to the opposite angle of the quadrilateral.

Ex. 2. If the sides AB, DC of a quadrilateral inscribed in a triangle be produced to meet in E, then the triangles EBC, EAD will be equiangular.

Ex. 3. Shew that a circle cannot be described about a rhombus.

Ex. 4. The lines, bisecting any angle of a quadrilateral figure inscribed in a circle and the opposite exterior angle, meet in the circumference of the circle.

Ex. 5. AB, a chord of a circle, is the base of an isosceles triangle, whose vertex C is without the circle, and whose equal sides meet the circle in D, E: shew that CD is equal to CE.

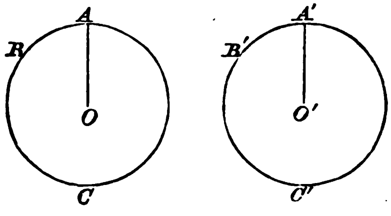
Ex. 6. If in any quadrilateral the opposite angles be together equal to two right angles, a circle may be described about that quadrilateral.

Propositions XXIII and XXIV., not being required in the method adopted for proving the subsequent propositions in this book, are removed to the Appendix. Proposition XXV. has been already proved.

NOTE III. *On the Method of Superposition, as applied to Circles.*

In Props. XXVI. XXVII. XXVIII. XXIX. we prove certain relations existing between chords, arcs and angles in equal circles. As we shall employ the Method of Superposition, we must state the principles which render this method *applicable*, as a test of equality, in the case of figures with *circular* boundaries.

Def. XIII. *Equal circles are those, of which the radii are equal.*



For suppose ABC, $A'B'C$ to be circles, of which the radii are equal.

Then if ⊙ $A'B'C'$ be applied to ⊙ ABC, so that O', the centre of $A'B'C'$, coincides with O, the centre of ABC, it is evident that any *particular* point A' in the ○ce of the former must coincide with *some* point A in the ○ce of the latter, because of the equality of the radii $O'A'$ and OA.

Hence ○ce $A'B'C'$ must coincide with ○ce ABC,

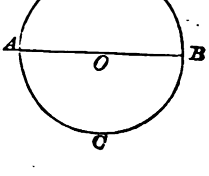
that is, ⊙ $A'B'C'$ = ⊙ ABC.

Further, when we have applied the circle $A'B'C'$ to the circle ABC, so that the centres coincide, we may imagine ABC to remain fixed while $A'B'C'$ revolves round the common centre. Hence we may suppose any particular point B' in the circumference of $A'B'C'$ to be made to coincide with any particular point B in the circumference of ABC.

Again, any radius $O'A'$ of the circle $A'B'C'$ may be made to coincide with any radius OA of the circle ABC.

Also, if $A'B'$ and AB be equal arcs, they may be made to coincide.

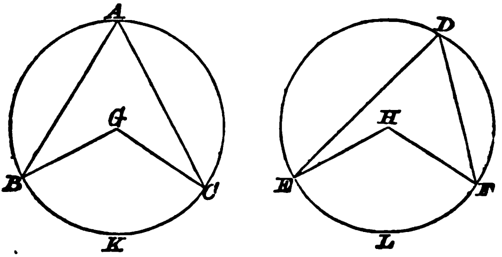
Again, every diameter of a circle divides the circle into equal segments.



For let AOB be a diameter of the circle $ACBD$, of which O is the centre. Suppose the segment ACB to be applied to the segment ADB, so as to keep AB a common boundary: then the arc ACB must coincide *with the arc* ADB, because every *point in each is* equally distant from O.

PROPOSITION XXVI. THEOREM.

In equal circles, the arcs, which subtend equal angles, whether they be at the centres or at the circumferences, must be equal.



Let ABC, DEF be equal circles, and let ∠s BGC, EHF at their centres, and ∠s BAC, EDF at their ○ces, be equal.

Then must arc BKC = *arc* ELF.

For, if ⊙ ABC be applied to ⊙ DEF,

so that G coincides with H, and GB falls on HE,

then, ∵ $GB = HE$, ∴ B will coincide with E.

And ∵ $\angle BGC = \angle EHF$, ∴ GC will fall on HF;

and ∵ $GC = HF$, ∴ C will coincide with F.

Then ∵ B coincides with E and C with F,

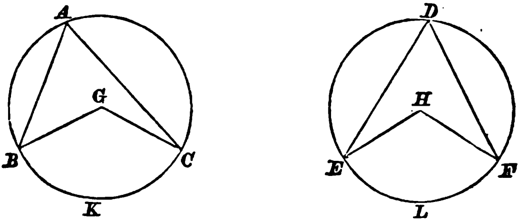
∴ arc BKC will coincide with and be equal to arc ELF,

Q. E. D.

NOTE. This and the three following Propositions are, and will *hereafter be* assumed to be, true for the same circle as *well as for equal circles.*

PROPOSITION XXVII. THEOREM.

In equal circles, the angles, which are subtended by equal arcs, whether they are at the centres or at the circumferences must be equal.



Let *ABC*, *DEF* be equal circles, and let ∠s *BGC*, *EHF* at their centres, and ∠s *BAC*, *EDF* at their ○ces, be subtended by equal arcs *BKC*, *ELF*.

Then must ∠ BGC = ∠ EHF, *and* ∠ BAC = ∠ EDF.

For, if ⊙ *ABC* be applied to ⊙ *DEF*,
so that *G* coincides with *H*, and *GB* falls on *HE*,
then ∵ *GB* = *HE*, ∴ *B* will coincide with *E*;
and ∵ arc *BKC* = arc *ELF*, ∴ *C* will coincide with *F*.
Hence, *GC* will coincide with *HF*.
Then ∵ *BG* coincides with *EH*, and *GC* with *HF*,
∴ ∠ *BGC* will coincide with and be equal to ∠ *EHF*.
Again, ∵ ∠ *BAC* = half of ∠ *BGC*, III. 20.
and ∠ *EDF* = half of ∠ *EHF*, III. 20.
∴ ∠ *BAC* = ∠ *EDF*. I. Ax. 7.
Q.E.D.

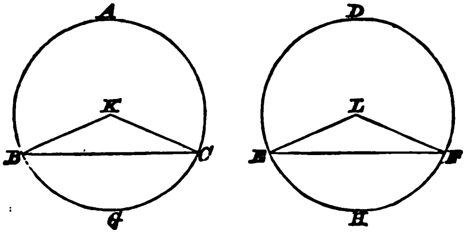
Ex. 1. If, in a circle, *AB*, *CD* be two arcs of given magnitude, and *AC*, *BD* be joined to meet in *E*, shew that the angle *AEB* is invariable.

Ex. 2. The straight lines joining the extremities of the chords of two equal arcs of the same circle, towards the same parts, are parallel to each other.

Ex. 3. If two equal chords, in a given circle, cut one *another, the segments* of the one shall be equal to the segments *of the other,* each to each.

PROPOSITION XXVIII. THEOREM.

In equal circles, the arcs, which are subtended by equal chords, must be equal, the greater to the greater, and the less to the less.



Let *ABC*, *DEF* be equal circles, and *BC*, *EF* equal chords, subtending the major arcs *BAC*, *EDF*,

and the minor arcs *BGC*, *EHF*.

Then must arc BAC = *arc* EDF, *and arc* BGC = *arc* EHF.

Take the centres *K*, *L*, and join *KB*, *KC*, *LE*, *LF*.

Then $\because KB = LE$, and $KC = LF$, and $BC = EF$,

$\therefore \angle BKC = \angle ELF$. I. C.

Hence, if ⊙ *ABC* be applied to ⊙ *DEF*,

so that *K* coincides with *L*, and *KB* falls on *LE*,

then $\because \angle BKC = \angle ELF$, $\therefore$ *KC* will fall on *LF*;

and $\because KC = LF$, $\therefore$ *C* will coincide with *F*.

Then $\because$ *B* coincides with *E*, and *C* with *F*,

$\therefore$ arc *BAC* will coincide with and be equal to arc *EDF*,

and arc *BGC* .... *EHF*.

Q.E.D.

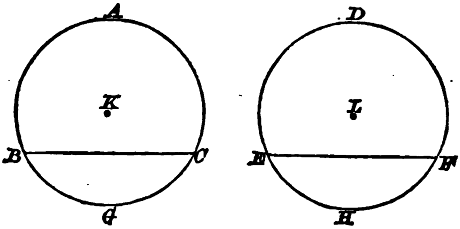
Ex. 1. If, in a circle *ABCD*, the arc *AB* be equal to arc *DC*, *AD* must be parallel to *BC*.

Ex. 2. If a straight line, drawn from *A* the middle point of an arc *BC*, touch the circle, shew that it is parallel to the chord *BC*.

Ex. 3. If two chords of a circle intersect at right angles, *the portions* of the circumference taken alternately are *together* equal to half the circumference.

PROPOSITION XXIX. THEOREM.

In equal circles, the chords, which subtend equal arcs, must be equal.



Let *ABC*, *DEF* be equal circles, and let *BC*, *EF* be chords, subtending the equal arcs *BGC*, *EHF*.

Then must chord BC = *chord* EF.

Take the centres *K*, *L*.

Then, if ⊙ *ABC* be applied to ⊙ *DEF*,

so that *K* coincides with *L*, and *B* with *E*,

and arc *BGC* falls on arc *EHF*,

∵ arc *BGC*=arc *EHF*, ∴ *C* will coincide with *F*.

Then ∵ *B* coincides with *E* and *C* with *F*,

∴ chord *BC* must coincide with and be equal to chord *EF*.

Q. E. D.

Ex. 1. The two straight lines in a circle, which join the extremities of two parallel chords, are equal to one another.

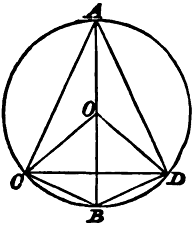
Ex. 2. If three equal chords of a circle cut one another *in the same point,* within the circle, that point is the centre.

NOTE IV. ***On the Symmetrical properties of the Circle with regard to its diameter.***

The brief remarks on Symmetry in pp. 107, 108 of Part I. may now be extended in the following way:

A *figure* is said to be symmetrical with regard to a line, when every perpendicular to the line meets the figure at points, which are equidistant from the line.

Hence a Circle is Symmetrical with regard to its Diameter, because the diameter *bisects* every chord, to which it is perpendicular.



Further, suppose AB to be a diameter of the circle $ACBD$, of which O is the centre, and CD to be a chord perpendicular to AB.

Then, if lines be drawn as in the diagram, we know that AB bisects

(1) The chord CD,

(2) The arcs CAD and CBD,

(3) The angles CAD, COD, CBD, and the reflex angle DOC.

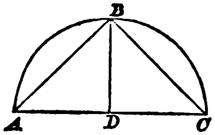
Also, chord CB = chord DB,

and chord AC = chord AD.

These Symmetrical relations should be carefully observed, *because they are often suggestive of methods for the solution of problems.*

PROPOSITION XXX. PROBLEM.

To bisect a given arc.



Let ABC be the given arc.

It is required to bisect the arc ABC.

Join AC, and bisect the chord AC in D. I. 10.

From D draw $DB \perp$ to AC. I. 11.

Then will the arc ABC *be bisected in* B.

Join BA, BC.

Then, in $\triangle$s ADB, CDB,

$\because AD = CD$, and DB is common, and $\angle ADB = \angle CDB$,

$\therefore BA = BC$. I. 4.

But, in the same circle, the arcs, which subtend equal chords, are equal, the greater to the greater and the less to the less; III. 28.

and $\because BD$, if produced, is a diameter,

$\therefore$ each of the arcs BA, BC, is less than a semicircle,

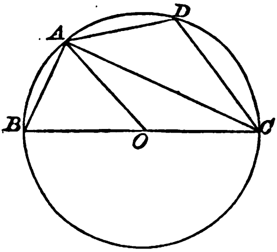
and $\therefore$ arc BA = arc BC.

Thus the arc ABC is bisected in B. Q. E. F.

Ex. If, from any point in the diameter of a semicircle, there be drawn two straight lines to the circumference, one to the bisection of the circumference, and the other at right *angles to the diameter,* the squares on these two lines are *together double of the* square on the radius.

PROPOSITION XXXI. THEOREM.

In a circle, the angle in a semicircle is a right angle; and the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.



Let ABC be a ⊙, O its centre, and BC a diameter.

Draw AC, dividing the ⊙ into the segments ABC, ADC.

Join BA, AD, DC, AO.

Then must the $\angle$ *in the semicircle* BAC *be a rt.* $\angle$*, and* $\angle$ *in segment* ABC, *greater than a semicircle, less than a rt.* $\angle$*, and* $\angle$ *in segment* ADC, *less than a semicircle, greater than a rt.* $\angle$.

First, $\because BO=AO$, $\therefore \angle BAO = \angle ABO$;

$\therefore \angle COA = $ twice $\angle BAO$; I. 32.

and $\because CO=AO$, $\therefore \angle CAO = \angle ACO$;

$\therefore \angle BOA = $ twice $\angle CAO$; I. 32.

$\therefore$ sum of $\angle$s COA, $BOA =$ twice sum of $\angle$s BAO, CAO, that is, two right angles $=$ twice $\angle BAC$.

$\therefore \angle BAC$ is a right angle.

Next, $\because \angle BAC$ is a rt. $\angle$,

$\therefore \angle ABC$ is less than a rt. $\angle$. I. 17.

Lastly, $\because$ sum of $\angle$s ABC, $ADC =$ two rt. $\angle$s, III. 22.

and $\angle ABC$ is less than a rt. $\angle$,

$\therefore \angle ADC$ is greater than a rt. $\angle$. Q. E. D.

NOTE. For a simpler proof see page 210.

Ex. 1. If a circle be described on the radius of anotl circle as diameter, any straight line, drawn from the poi where they meet, to the outer circumference, is bisected the interior one.

Ex. 2. If a straight line be drawn to touch a circle, ε be parallel to a chord, the point of contact will be the mid point of the arc cut off by the chord.

Ex. 3. If, from any point without a circle, lines be dra touching it, the angles contained by the tangents is dou of the angle contained by the line joining the points of cont and the diameter drawn through one of them.

Ex. 4. The vertical angle of any oblique-angled trian inscribed in a circle is greater or less than a right angle, the angle contained by the base and the diameter drawn fr the extremity of the base.

Ex. 5. If, from the extremities of any diameter of a gi circle, perpendiculars be drawn to any chord of the ci that is not parallel to the diameter, the less perpendicu shall be equal to that segment of the greater, which contained between the circumference and the chord.

Ex. 6. If two circles cut one another, and from eitl point of intersection diameters be drawn, the extremities these diameters and the other point of intersection lie in same straight line.

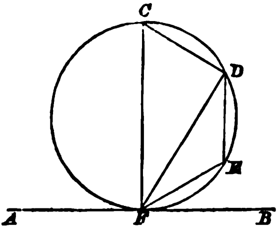
Ex. 7. Draw a straight line cutting two concentric circ so that the part of it which is intercepted by the circumfere of the greater may be twice the part intercepted by the cumference of the less.

Ex. 8. Describe a square equal to the difference of given squares.

Ex. 9. If from the point in which a number of cir touch each other, a straight line be drawn cutting all circles, shew that the lines, which join the points of in section in each circle with its centre, will all be parallel.

PROPOSITION XXXII. THEOREM.

If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles made by this line with the line touching the circle must be equal to the angles, which are in the alternate segments of the circle.



Let the st. line *AB* touch the ⊙ *CDEF* in *F*.

Draw the chord *FD*, dividing the ⊙ into segments *FCD*, *FED*.

Then must ∠ DFB = ∠ *in segment* FCD,

and ∠ DFA = ∠ *in segment* FED.

From *F* draw the chord *FC* ⊥ to *AB*.

Then *FC* is a diameter of the ⊙. III. 19.

Take any pt. *E* in the arc *FED*, and join *FE*, *ED*, *DC*.

Then ∵ *FDC* is a semicircle, ∴ ∠ *FDC* is a rt. ∠; III. 31.

∴ sum of ∠s *FCD*, *CFD* = a rt. ∠. I. 32.

Also, sum of ∠s *DFB*, *CFD* = a rt. ∠;

∴ sum of ∠s *DFB*, *CFD* = sum of ∠s *FCD*, *CFD*,

and ∴ ∠ *DFB* = ∠ *FCD*,

that is, ∠ *DFB* = ∠ in segment *FCD*.

Again, ∵ *CDEF* is a quadrilateral fig. inscribed in a ⊙,

∴ sum of ∠s *FED*, *FCD* = two rt. ∠s. III. 22.

Also, sum of ∠s *DFA*, *DFB* = two rt. ∠s; I. 13.

∴ sum of ∠s *DFA*, *DFB* = sum of ∠s *FED*, *FCD*;

and ∠ *DFB* has been proved = ∠ *FCD*;

∴ ∠ *DFA* = ∠ *FED*,

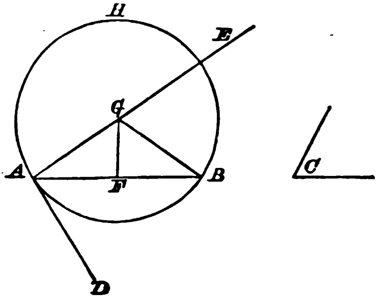
that is, ∠ *DFA* = ∠ in segment *FED*.

Q. E. D.

Ex. The chord joining the points of contact of parallel ngents is a diameter.

PROPOSITION XXXIII. PROBLEM.

On a given straight line to describe a segment of a circle containing an angle equal to a given angle.



Let AB be the given st. line, and C the given ∠.

It is reqd. to describe on AB *a segment of a* ⊙ *which shall contain an* ∠ = ∠ C.

At pt. A in st. line AB make ∠ BAD = ∠ C.

Draw AE ⊥ to AD, and bisect AB in F.

From F draw FG ⊥ to AB, meeting AE in G.

Then in △s AGF, BGF;

∴ $AF = BF$, and FG is common, and ∠ AFG = ∠ BFG;

∴ $GA = GB$. I. 4.

With G as centre and GA as radius describe a ⊙ ABH.

Then will AHB be the segment reqd.

For ∵ AD is ⊥ to AE, a line passing through the centre,

∴ AD is a tangent to the ⊙ ABH. III. 16.

And ∵ the chord AB is drawn from the pt. of contact A,

∴ ∠ BAD = ∠ in segment AHB, III. 32.

that is, the segment AHB contains an ∠ = ∠ C,

and it is described on AB, as was reqd.

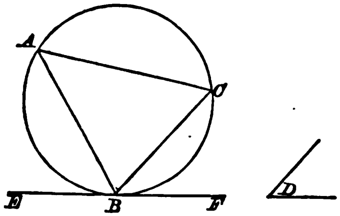
Q. E. F.

Ex. 1. Two circles intersect in A, and through A is drawn a straight line meeting the circles again in P, Q. Prove that the angle between the tangents at P and Q is equal to the angle between the tangents at A.

Ex. 2. From two given points on the same side of a straight line, given in position, draw two straight lines which shall contain a given angle, and be terminated in the given line.

PROPOSITION XXXIV. PROBLEM.

To cut off a segment from a given circle, capable of containing an angle equal to a given angle.



Let ABC be the given ⊙, and D the given ∠.

It is reqd. to cut off from ⊙ ABC *a segment capable of containing an* ∠ = ∠ D.

Draw the st. line EBF to touch the circle at B.

At B make ∠ FBC = ∠ D.

Then ∵ the chord BC is drawn from the pt. of contact B,

∴ ∠ FBC = ∠ in segment BAC, III. 32.

that is, the segment BAC contains an ∠ = ∠ D;

and ∴ a segment has been cut off from the ⊙ as was reqd.

Q. E. F.

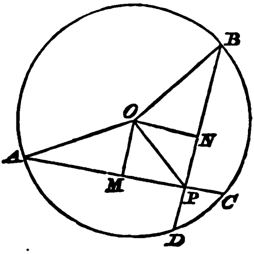
Ex. 1. If two circles touch internally at a point, any straight line passing through the point will divide the circles into segments, capable of containing equal angles.

Ex. 2. Given a side of a triangle, its vertical angle, and the radius of the circumscribing circle: construct the triangle.

Ex. 3. Given the base, vertical angle, and the perpendicular from the extremity of the base on the opposite side: construct the triangle.

PROPOSITION XXXV. THEOREM.

If two chords in a circle cut one another, the rectangle contained by the segments of one of them, is equal to the rectangle contained by the segments of the other.



Let the chords AC, BD in the ⊙ $ABCD$, intersect in the pt. P.

Then must rect. AP, PC = *rect.* BP, PD.

From O, the centre, draw OM, ON ⊥s to AC, BD, and join OA, OB, OP.

Then ∵ AC is divided equally in M and unequally in P,

∴ rect. AP, PC with sq. on MP = sq. on AM. II. 5.

Adding to each the sq. on MO,

rect. AP, PC with sqq. on MP, MO = sqq. on AM, MO;

∴ rect. AP, PC with sq. on OP = sq. on OA. I. 47.

In the same way it may be shewn that

rect. BP, PD with sq. on OP = sq. on OB.

Then ∵ sq. on OA = sq. on OB,

∴ rect. AP, PC with sq. on OP = rect. BP, PD with sq. on OP;

∴ rect. AP, PC = rect. BP, PD.

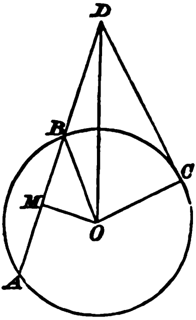
Q. E. D.

Ex. 1. A and B are fixed points, and two circles are described passing through them; PCQ, $P'CQ'$ are chords of these circles intersecting in C, a point in AB; shew that the rectangle CP, CQ is equal to the rectangle CP', CQ'.

Ex. 2. If through any point in the common chord of two circles, which intersect one another, there be drawn any two other chords, one in each circle, their four extremities shall all lie in the circumference of a circle.

PROPOSITION XXXVI. THEOREM.

If, from any point without a circle, two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, must be equal to the square on the line which touches it.



Let D be any pt. without the ⊙ ABC,

and let the st. lines DBA, DC be drawn to cut and touch the ⊙.

Then must rect. AD, DB = *sq. on* DC.

From O, the centre, draw $OM \perp$ to AB,

and join OB, OC, OD.

Then $\because AB$ is bisected in M and produced to D,

$\therefore$ rect. AD, DB with sq. on $MB =$ sq. on MD. II. 6.

Adding to each the sq. on MO,

rect. AD, DB with sqq. on MB, $MO =$ sqq. on MD, MO.

Now the angles at M and C are rt. $\angle$s; III. 3 and 17.

$\therefore$ rect. AD, DB with sq. on $OB =$ sq. on OD;

$\therefore$ rect. AD, DB with sq. on $OB =$ sqq. on OC, DC. I. 47.

And sq. on $OB =$ sq. on OC;

$\therefore$ rect. AD, $DB =$ sq. on DC.

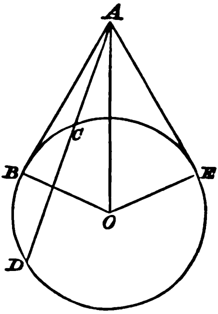
Q. E. D.

Ex. 1. Two circles intersect in A and B; shew that AB produced bisects their common tangent.

Ex. 2. If the circle, inscribed in a triangle ABC, touch BC in D, the circles described about ABD, ACD will touch each other.

PROPOSITION XXXVII. THEOREM.

If, from a point without a circle there be drawn tu straight lines, one of which cuts the circle, and the othe meets it; if the rectangle contained by the whole line whic cuts the circle, and the part of it without the circle, be equa to the square of the line which meets it, the line which meet must touch the circle.



Let A be a pt. without the ⊙ BCD, of which O is the centre.

From A let two st. lines ACD, AB be drawn, of which ACD cuts the ⊙ and AB meets it.

Then if rect. DA, AC = *sq. on* AB, AB *must touch the* ⊙.

Draw AE touching the ⊙ in E, and join OB, OA, OE.

Then ∵ ACD cuts the ⊙, and AE touches it,
∴ rect. DA, AC = sq. on AE. III. 36.
But rect. DA, AC = sq. on AB; Hyp.
∴ sq. on AB = sq. on AE;
∴ $AB = AE$.
Then in the △s OAB, OAE,
∵ $OB = OE$, and OA is common, and $AB = AE$,
∴ ∠ ABO = ∠ AEO. I. C.
But ∠ AEO is a rt. ∠;
∴ ∠ ABO is a rt. ∠.
Now BO, if produced, is a diameter of the ⊙;
∴ AB touches the ⊙. III. 16.

Q. E. D.

Ex. If two circles cut each other, and from any point he *straight line* produced, which joins their intersections, *tangents be* drawn, one to each circle, they shall be equal.

Miscellaneous Exercises on Book III.

1. The segments into which a circle is cut by any straight line, contain angles whose difference is equal to the inclination to each other of the straight lines touching the circle at the extremities of the straight line which divides the circle.

2. If from the point in which a number of circles touch each other, a straight line be drawn cutting all the circles, shew that the lines which join the points of intersection in each circle with its centre will be all parallel.

3. From a point Q in a circle, QN is drawn perpendicular to a chord PP', and QM perpendicular to the tangent at P: shew that the triangles NQP', QPM are equiangular.

4. If a circle be described round the triangle ABC, and a straight line be drawn bisecting the angle BAC and cutting the circle in D, shew that the angle DCB will equal half the angle BAC.

5. One angle of a quadrilateral figure inscribed in a circle is a right angle, and from the centre of the circle perpendiculars are drawn to the sides, shew that the sum of their squares is equal to twice the square of the radius.

6. AB is the diameter of a semicircle, D and E any two points on its circumference. Shew that if the chords joining A and B with D and E, either way, intersect in F and G, the tangents at D and E meet in the middle point of the line FG, and that FG produced is at right angles to AB.

7. If a straight line in a circle not passing through the centre be bisected by another and this by a third and so on, prove that the points of bisection continually approach the centre of the circle.

8. If a circle be described passing through the opposite angles of a parallelogram, and cutting the four sides, and the points of intersection joined so as to form a hexagon, the straight lines thus drawn shall be parallel to each other.

9. If two circles touch each other externally and any third circle touch both, prove that the difference of the dis-

tances of the centre of the third circle from the centres of the other two is invariable.

10. Draw two concentric circles, such that those chords of the outer circle, which touch the inner, may equal its diameter.

11. If the sides of a quadrilateral inscribed in a circle be bisected and the middle points of adjacent sides joined, the circles described about the triangles thus formed are all equal and all touch the original circle.

12. Draw a tangent to a circle which shall be parallel to a given finite straight line.

13. Describe a circle, which shall have a given radius, and its centre in a given straight line, and shall also touch another straight line, inclined at a given angle to the former.

14. Find a point in the diameter produced of a given circle, from which, if a tangent be drawn to the circle, it shall be equal to a given straight line.

15. Two equal circles intersect in the points A, B, and through B a straight line CBM is drawn cutting them again in C, M. Shew that if with centre C and radius BM a circle be described, it will cut the circle ABC in a point L such that arc $AL =$ arc AB.

Shew also that LB is the tangent at B.

16. AB is any chord and AC a tangent to a circle at A; CDE a line cutting the circle in D and E and parallel to AB. Shew that the triangle ACD is equiangular to the triangle EAB.

17. Two equal circles cut one another in the points A, B; BC is a chord equal to AB; shew that AC is a tangent to the other circle.

18. In any two circles, which cut one another, the straight line joining the extremities of any two parallel radii cuts the line joining the centres in the same point.

19. A, B *are two* points; with centre B describe a circle, *such that its tangent* from A shall be equal to a given line.

20. If perpendiculars be dropped from the angular points of a triangle on the opposite sides, shew that the sum of the squares of the sides of the triangle is equal to twice the sum of the rectangles contained by the perpendiculars and that part of each intercepted between the angles of the triangles and the point of intersection of the perpendiculars.

21. When two circles intersect, their common chord bisects their common tangent.

22. Two circles intersect in A and B. Two points C and D are taken on one of the circles; CA, CB meet the other circle in E, F, and DA, DB meet it in G, H: shew that FG is parallel to EH, and FH to EG.

23. A and B are fixed points, and two circles are described passing through them; CP, CP' are drawn from a point C on AB produced, to touch the circles in P, P'; shew that $CP=CP'$.

24. From each angular point of a triangle a perpendicular is let fall upon the opposite side; prove that the rectangles, contained by the segments into which each perpendicular is divided by the point of intersection of the three, are equal to each other.

25. If from a point without a circle two equal straight lines be drawn to the circumference and produced, shew that they will be at the same distance from the centre.

26. Let O, O' be the centres of two circles which cut each other in A, A'. Let B, B' be two points, taken one on each circumference. Let C, C' be the centres of the circles BAB', $BA'B'$. Then prove that the angle CBC' is equal to the angle $OA'O'$.

27. The common chord of two circles is produced to any point P; PA touches one of the circles in A; PBC is any chord of the other: shew that the circle which passes through A, B, C touches the circle to which PA is a tangent.

28. Given the base of a triangle, the vertical angle, and the length of the line drawn from the vertex to the middle point of the base: construct the triangle.

29. *If a circle* be described about the triangle ABC, and *a straight line* be drawn bisecting the angle BAC and cutting

the circle in *D*, shew that the angle *DCB* will be equal to half the angle *BAC*.

30. If the line *AD* bisect the angle *A* in the triangle *ABC*, and *BD* be drawn without the triangle making an angle with *BC* equal to half the angle *BAC*, shew that a circle may be described about *ABCD*.

31. Two equal circles intersect in *A*, *B*: *PQT* perpendicular to *AB* meets it in *T* and the circles in *P*, *Q*. *AP*, *BQ* meet in *R*; *AQ*, *BP* in *S*; prove that the angle *RTS* is bisected by *TP*.

32. If the angle contained by any side of a quadrilateral and the adjacent side produced be equal to the opposite angle of the quadrilateral, prove that any side of the quadrilateral will subtend equal angles at the opposite angles of the quadrilateral.

33. If *DE* be drawn parallel to the base *BC* of a triangle *ABC*, prove that the circles described about the triangles *ABC* and *ADE* have a common tangent at *A*.

34. Describe a square equal to the difference of two given squares.

35. If tangents be drawn to a circle from any point without it, and a third line be drawn between the point and the centre of the circle, touching the circle, the perimeter of the triangle formed by the three tangents will be the same for all positions of the third point of contact.

36. If on the sides of any triangle as chords, circles be described, of which the segments external to the triangle contain angles respectively equal to the angles of a given triangle, those circles will intersect in a point.

37. Prove that if *ABC* be a triangle inscribed in a circle, such that $BA = BC$, and AA' be drawn parallel to *BC*, meeting the circle again in A', and $A'B$ be joined cutting *AC* in *E*, *BA* touches the circle described about the triangle AEA'.

38. Describe a circle, cutting the sides of a given square, so that its circumference may be divided at the points of intersection into eight equal arcs.

39. A is the extremity of the diameter of a circle, O any oint in the diameter. The chord which is bisected at O ubtends a greater or less angle at A than any other chord hrough O, according as O and A are on the same or opposite ides of the centre.

40. Shew that the square on the tangent drawn from any oint in the outer of two concentric circles to the inner quals the difference of the squares on the tangents, drawn om any point, without both circles, to the circles.

41. If from a point without a circle, two tangents PT T', at right angles to one another, be drawn to touch the rcle, and if from T any chord TQ be drawn, and from T' perpendicular $T'M$ be dropped on TQ, then $T'M = QM$.

42. Find the loci:

(1) Of the centres of circles passing through two given ints.

(2) Of the middle points of a system of parallel chords a circle.

(3) Of points such that the difference of the distances of h from two given straight lines is equal to a given straight e.

(4) Of the centres of circles touching a given line in a en point.

(5) Of the middle points of chords in a circle that pass ough a given point.

(6) Of the centres of circles of given radius which touch iven circle.

(7) Of the middle points of chords of equal length in a le.

(8) Of the middle points of the straight lines drawn from iven point to meet the circumference of a given circle.

43. If the base and vertical angle of a triangle be given, l the locus of the vertex.

44. A straight line remains parallel to itself while one of *extremities* describes a circle. What is the locus of the *r extremity ?*

45. A ladder slips down between a vertical wall and a horizontal plane: what is the locus of its middle point?

46. *AB* is the diameter of a circle; *ACD* is a chord produced to *D*, so that *AC*=*CD*. Find the locus of the point in which *BC* and the line joining *D* to the centre intersect.

47. *ABC* is a line drawn from a point *A*, without a circle, to meet the circumference in *B* and *C*. Tangents are drawn to the circle at *B* and *C* which meet in *D*. What is the locus of *D*?

48. Two circles intersect in the points *A*, *B*; any straight line *CDEF* is drawn cutting the circles in *C*, *D*, *E*, *F*; prove that *AC* intersects *BD* and *AE* intersects *BF* in points which lie on a circle passing through *A* and *B*.

49. The angular points *A*, *C* of a parallelogram *ABCD* move on two fixed straight lines *OA*, *OC*, whose inclination is equal to the angle *BCD*; shew that the points *B*, *D* will move on two fixed straight lines passing through *O*.

50. On the line *AB* is described the segment of a circle, in the circumference of which any point *C* is taken If *AC*, *BC* be joined, and a point *P* taken in *AC* so that *CP* is equal to *CB*, find the locus of *P*.

51. Find the locus of the centre of the circles circumscribing two trapeziums into which a parallelogram is divided by any line equal to one of its shorter sides.

52. If a parallelogram be described having the diameter of a given circle for one of its sides, and the intersection of its diagonals on the circumference, shew that the extremity of each of the diagonals moves on the circumference of another circle of double the diameter of the first.

53. One diagonal of a quadrilateral inscribed in a circle is fixed, and the other of constant length. Shew that the sides will meet if produced on the circumference of a fixed circle.

BOOK IV.

INTRODUCTORY REMARKS.

Euclid gives in this Book of the Elements a series of Problems relating to cases in which circles may be described in or about triangles, squares, and regular polygons, and of the last-mentioned he treats of three only:

the Pentagon, or figure of 5 sides,
„ Hexagon, „ 6 „
„ Quindecagon „ 15 „ .

The student will find it useful to remember the following Theorems, which are established and applied in the proofs of the Propositions in this Book.

I. The bisectors of the angles of a triangle, square, or regular polygon meet in a point, which is the centre of the inscribed circle.

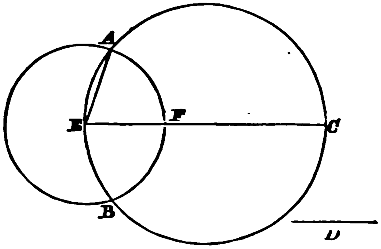
II. The perpendiculars drawn from the middle points of the sides of a triangle, square, or regular polygon meet in a point, which is the centre of the circumscribed circle.

III. In the case of a square, or regular polygon the inscribed and circumscribed circles have a common centre.

IV. If the circumference of a circle be divided into any number of equal parts, the chords joining each pair of consecutive points form a regular figure inscribed in the circle, and the tangents drawn through the points form a regular figure described about the circle.

Proposition I. Problem.

In a given circle to place a straight line equal to a gi straight line, which is not greater than the diameter of t circle.



Let ABC be the given ⊙, and D the given line, not greater than the diameter of the ⊙.

It is required to place in the ⊙ ABC *a st. line* = D.

Draw EC, a diameter of ⊙ ABC.

Then if $EC = D$, what was required is done.

But if not, EC is greater than D. From EC cut off $EF = D$, and with centre E and radius EF describe a ⊙ AFB, cutting the ⊙ABC in A and B; and join AE.

Then, $\because$ E is the centre of ⊙ AFB,

$\therefore EA = EF$,

and $\therefore EA = D$.

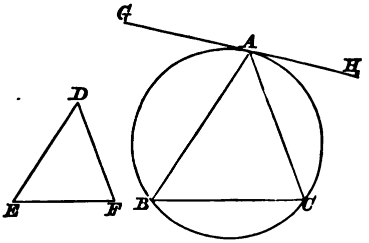
Thus a st. line EA equal to D has been placed in ⊙ ABC.

Q. E. F.

Ex. Draw the diameter of a circle, which shall pass at *iven distance from* a given point.

PROPOSITION II. PROBLEM.

In a given circle to inscribe a triangle, equiangular to a given triangle.



Let ABC be the given ⊙, and DEF the given △.

It is required to inscribe in ⊙ ABC *a* △, *equiangular to* △ DEF.

Draw GAH touching the ⊙ ABC at the pt. A.

Make $\angle GAB = \angle DFE$, and $\angle HAC = \angle DEF$.

Join BC. Then will △ ABC be the required △.

For ∵ GAH is a tangent, and AB a chord of the ⊙,

∴ $\angle ACB = \angle GAB$, III. 32.

that is, $\angle ACB = \angle DFE$.

So also, $\angle ABC = \angle HAC$, III. 32.

that is, $\angle ABC = \angle DEF$;

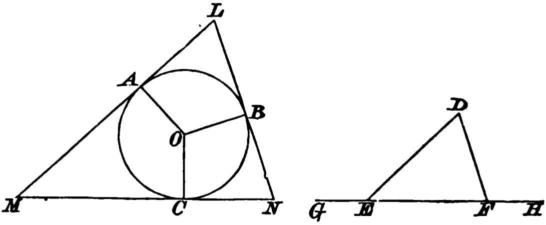
∴ remaining $\angle BAC$ = remaining $\angle EDF$;

∴ △ ABC is equiangular to △ DEF, and it is inscribed in the ⊙ ABC. Q. E. F.

Ex. If an equilateral triangle be inscribed in a circle, *prove that the* radii, drawn to the angular points, bisect the *angles of the* triangle.

PROPOSITION III. PROBLEM.

About a given circle to describe a triangle, equiangul to a given triangle.



Let ABC be the given ⊙, and DEF the given △.

It is required to describe about the ⊙ a △, equiangu to △ EDF.

From O, the centre of the ⊙, draw any radius OC.

Produce EF to the pts. G, H.

Make $\angle AOC = \angle DEG$, and $\angle BOC = \angle DFH$.

Through A, B, C draw tangents to the ⊙, meeting in L, M,

Then will LMN be the △ required.

For $\because$ ML, LN, NM are tangents to the ⊙,

$\therefore$ the $\angle$s at A, B, C are rt. $\angle$s. III.

Now $\angle$s of quadrilateral $AOCM$ together = four rt. $\angle$

and of these $\angle OAM$ and $\angle OCM$ are rt. $\angle$s;

$\therefore$ sum of $\angle$s AOC, AMC = two rt. $\angle$s.

But sum of $\angle$s DEG, DEF = two rt. $\angle$s; I.

$\therefore$ sum of $\angle$s AOC, AMC = sum of $\angle$s DEG, DEF,

and $\angle AOC = \angle DEG$, by construction;

$\therefore$ $\angle AMC = \angle DEF$;

that is, $\angle LMN = \angle DEF$.

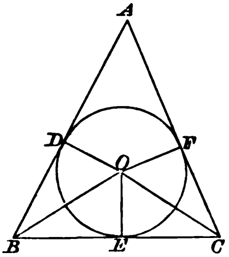
Similarly, it may be shewn that $\angle LNM = \angle DFE$;

$\therefore$ also $\angle MLN = \angle EDF$.

Thus a △, equiangular to △ DEF, is described ab the ⊙. Q

PROPOSITION IV. PROBLEM.

To inscribe a circle in a given triangle.



Let ABC be the given △.

It is required to inscribe a ⊙ *in the* △ ABC.

Bisect ∠s ABC, ACB by the st. lines BO, CO, meeting in O.

From O draw OD, OE, OF, ⊥s to AB, BC, CA.

Then, in △s EBO, DBO,

∵ ∠EBO = ∠DBO, and ∠BEO = ∠BDO, and OB is common,

∴ $OE = OD$. I. 26.

Similarly it may be shewn that $OE = OF$.

If then a ⊙ be described, with centre O, and radius OD, this ⊙ will pass through the pts. D, E, F;

and ∵ the ∠s at D, E and F are rt. ∠s,

∴ AB, BC, CA are tangents to the ⊙; III. 16.

and thus a ⊙ DEF may be inscribed in the △ ABC.

Q. E. F.

Ex. 1. Shew that, if OA be drawn, it will bisect the angle BAC.

Ex. 2. If a circle be inscribed in a right-angled triangle, the difference between the hypotenuse and the sum of the other sides is equal to the diameter of the circle.

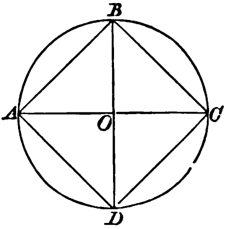
Ex. 3. Shew that, in an equilateral triangle, the centre of the inscribed circle is equidistant from the three angular points.

Ex. 4. Describe a circle, touching one side of a triangle and the other two produced. (Note. This is called an *escribed* circle.)

Note. Euclid's fifth Proposition of this Book has
already given on page 15.

PROPOSITION VI. PROBLEM.

To inscribe a square in a given circle.



Let *ABCD* be the given ⊙.

It is required to inscribe a square in the ⊙.

Through *O*, the centre, draw the diameters *AC*, *BD*,
each other.

Join *AB*, *BC*, *CD*, *DA*.

Then ∵ the ∠s at *O* are all equal, being rt. ∠s,

∴ the arcs *AB*, *BC*, *CD*, *DA* are all equal, III

and ∴ the chords *AB*, *BC*, *CD*, *DA* are all equal; III

and ∠ *ABC*, being the ∠ in a semicircle, is a rt. ∠. III

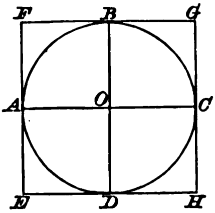
So also the ∠s *BCD*, *CDA*, *DAB* are rt. ∠s;

∴ *ABCD* is a square,

and it is inscribed in the ⊙ as was required.

PROPOSITION VII. PROBLEM.

To describe a square about a given circle.



Let $ABCD$ be the given ⊙, of which O is the centre.

It is required to describe a square about the ⊙.

Draw the diameters AC, BD, $\perp$ to each other.

Through A, B, C D draw EF, FG, GH, HE touching the ⊙.

Then the $\angle$s at A, B, C, D are rt. $\angle$s. III. 16.

Now $\because$ the $\angle$s at A, O, C are all rt. $\angle$s,

$\therefore$ FE, BD, and GH are all $\parallel$; I. 27.

and $\because$ the $\angle$s at B, O, D are all rt. $\angle$s,

$\therefore$ FG, AC, and EH are all $\parallel$;

$\therefore$ FE and GH each $=BD$, I 34.

and FG and EH each $=AC$, I. 34.

And $\because$ $BD=AC$,

$\therefore$ FE, GH, FG, EH are all equal.

Again, $\because$ FO is a ▱,

$\therefore$ $\angle AFB = \angle AOB$, I. 34.

and $\therefore$ $\angle AFB$ is a rt. $\angle$.

So also the $\angle$s at G, H and E are rt. $\angle$s.

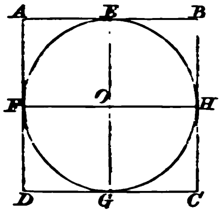
Hence $EFGH$ is a square, and it is described about the ⊙.

Q. E. F.

Ex. In a given circle inscribe four circles, equal to each *other, and in mutual contact with each other and with the given circle.*

PROPOSITION VIII. PROBLEM.

To inscribe a circle in a given square.



Let $ABCD$ be the given square.

It is required to inscribe a ⊙ in the square.

Bisect AB, AD in E, F,

and draw $EG \parallel$ to AD or BC, and $FH \parallel$ to AB or DC

Let EG and FH intersect in O.

Then ∵ AO is a ▱,

∴ $OE = FA$ and $OF = EA$. I.

But ∵ $AB = AD$, and E, F are the middle pts. of AB, $\angle$

∴ $FA = EA$,

and ∴ $OE = OF$.

Similarly, it may be shewn that $OG = OF$, and $OH = O$

and ∴ OE, OF, OG, OH are all equal ;

and a ⊙, described with centre O and radius OE,

will pass through E, F, G, H,

and it will be touched by each of the sides of the squar

∵ the $\angle$s at E, F, G, H are rt. $\angle$s. III.

Thus a ⊙ $EFGH$ may be described in the sq. $ABCD$.

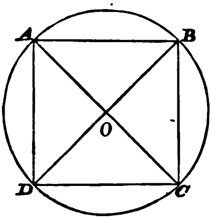
Q.

Ex. 1. In what parallelograms can circles be inscribed

Ex. 2. If, from any point in the circumference of a ci straight lines be drawn to the angular points of the inscri *square, the sum of* the squares on these four lines wi *double of the* square on the diameter.

PROPOSITION IX. PROBLEM.

To describe a circle about a given square.



Let $ABCD$ be the given square.

It is required to describe a ⊙ about the square.

Draw the diagonals AC, BD, intersecting each other in O.

Then $\because \angle DAC = \angle ACD$, I. A.

and $\angle BAC =$ alternate $\angle ACD$, I. 29.

$\therefore \angle DAC = \angle BAC$.

Thus the diagonal AC bisects $\angle BAD$,

and $\therefore \angle OAB =$ half a rt. $\angle$.

Similarly it may be shewn that $\angle OBA =$ half a rt. $\angle$;

$\therefore \angle OAB = \angle OBA$;

$\therefore OA = OB$. I. B. Cor.

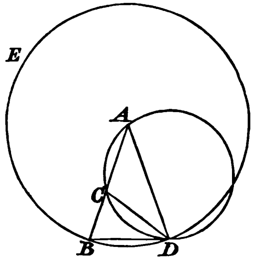
Similarly it may be shewn that $OC = OB$, and $OD = OA$;

$\therefore OA, OB, OC, OD$ are all equal;

and $\therefore$ a ⊙, described with centre O and radius OA, will pass through A, B, C, D, and will be described about the square, *as was required.* Q. E. F.

PROPOSITION X. PROBLEM.

To describe an isosceles triangle, having each of the angles at the base double of the third angle.



Take any st. line AB and divide it in C,

so that rect. AB, BC = sq. on AC. II. 11.

With centre A and radius AB describe the $\odot$ BDE,

and in it place the st. line $BD = AC$; and join AD.

Then will $\triangle ABD$ have each of the $\angle$s at the base double of $\angle BAD$.

Join CD, and about the $\triangle ACD$ describe the $\odot$ ACD.

Then $\because$ rect. AB, BC = sq. on AC, and $BD = AC$,

$\therefore$ rect. AB, BC = sq. on BD,

and $\therefore$ BD touches the $\odot$ ACD. III. 37.

Then $\because$ BD touches $\odot$ ACD, and DC is a chord of the $\odot$,

$\therefore$ $\angle BDC = \angle CAD$. III. 32.

Add to each $\angle CDA$.

Then $\angle BDA$ = sum of $\angle$s CAD, CDA,

$\therefore$ $\angle BDA = \angle BCD$. I. 32.

But $\angle BDA = \angle CBD$; I. A.

$\therefore$ $\angle BCD = \angle CBD$,

and $\therefore$ $BD = CD$.

But $BD = CA$;

$\therefore$ $CA = CD$,

and $\therefore$ $\angle CDA = \angle CAD$.

Hence sum of $\angle$s CDA, CAD = twice $\angle CAD$,

$\therefore$ $\angle BCD$ = twice $\angle BAD$. I. 32.

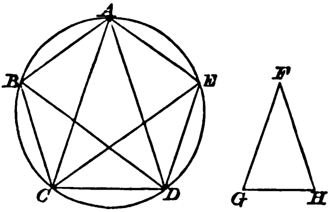
But $\angle ABD$ and $\angle ADB$ are each = $\angle BCD$,

$\therefore$ $\angle ABD$ and $\angle ADB$ are each = twice $\angle BAD$;

and thus an isosceles $\triangle$ ABD has been described as was *quired.* Q. E. F.

PROPOSITION XI. PROBLEM.

To inscribe a regular pentagon in a given circle.



Let $ABCDE$ be the given ⊙.

It is required to inscribe a regular pentagon in the ⊙.

Make an isosceles △ FGH, having each of the ∠ s at G, H double of ∠ at F.

In ⊙ $ABCDE$ inscribe a △ ACD equiangular to △ FGH, IV. 2.
having ∠ s at A, D, C = the ∠ s at F, G, H, respectively.
Then ∠ ADC = twice ∠ DAC, and ∠ ACD = twice ∠ DAC.

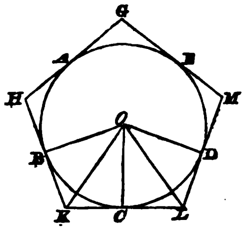
Bisect the ∠ s ADC, ACD by the chords DB, CE.
Join AB, BC, CD, DE, EA.
Then will $ABCDE$ be a regular pentagon.
For ∵ ∠ s ADC, ACD are each = twice ∠ DAC,
and ∠ s ADC, ACD are bisected by DB, CE,
∴ ∠ s ADB, BDC, DAC, ECD, ACE, are all equal;
and ∴ arcs AB, BC, CD, DE, EA are all equal; III. 26.
and ∴ chords AB, BC, CD, DE, EA are all equal. III. 29.
Hence, the pentagon $ABCDE$ is equilateral.
Again, ∵ arc CD = arc AB,
adding to each arc AED, we have
arc $AEDC$ = arc $BAED$,
and ∴ ∠ ABC = ∠ BCD. III. 27.
Similarly, ∠ s CDE, DEA, EAB each = ∠ ABC.
Hence, the pentagon $ABCDE$ is equiangular.
Thus a regular pentagon has been inscribed in the ⊙.

Q. E. F.

Ex. *Shew that* CE is parallel to BA.

PROPOSITION XII. PROBLEM.

To describe a regular pentagon about a given circle.



Let $ABCDE$ be the given ⊙,

It is required to describe a regular pentagon about the ⊙.

Let the angular pts. of a pentagon inscribed in the ⊙ be at $A, B, C, D, E,$

so that arcs AB, BC, CD, DE, EA are all equal.

Through A, B, C, D, E draw GH, HK, KL, LM, MG tangents to the ⊙;

find the centre O, and join OB, OK, OC, OL, OD.

Then in △s OBK, OCK,

∵ $OB = OC$, and OK is common, and $KB = KC$,
III. 37, Cor.

∴ ∠ $BKO =$ ∠ CKO, and ∠ $BOK =$ ∠ COK,

that is, ∠ $BKC =$ twice ∠ CKO, and ∠ $BOC =$ twice ∠ COK.

So also, ∠ $DLC =$ twice ∠ CLO, and ∠ DOC = twice ∠ COL.

Now $\because$ arc $BC=$ arc CD,

$\therefore \angle BOC = \angle DOC$,

and $\therefore \angle COK = \angle COL$.

Hence in $\triangle$s OCK, OCL,

$\because \angle COK = \angle COL$, and rt. $\angle OCK =$ rt. $\angle OCL$, and OC is common,

$\therefore \angle CKO = \angle CLO$, and $CK = CL$, I. B.

and $\therefore \angle HKL = \angle MLK$, and $KL =$ twice KC.

Similarly it may be shewn that $\angle$s KHG, HGM, GML each $= \angle HKL$,

$\therefore$ the pentagon $GHKLM$ is equiangular.

And since it has been shewn that $KL =$ twice KC,

and it can be shewn that $HK =$ twice KB,

and $\because KB = KC$,

$\therefore HK = KL$.

In like manner it may be shewn that HG, GM, ML, each $= KC$,

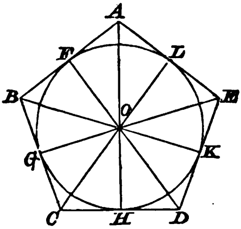
$\therefore$ the pentagon $GHKLM$ is equilateral.

Thus a regular pentagon has been described about the $\odot$.

Q. E. F.

PROPOSITION XIII. PROBLEM.

To inscribe a circle in a given regular pentagon.



Let $ABCDE$ be the given regular pentagon.

It is required to inscribe a ⊙ in the pentagon.

Bisect $\angle$s BCD, CDE by the st. lines CO, DO, meeting in
Join OB, OA, OE.

Then, in △s BCO, DCO,

$\because BC = DC$, and CO is common, and $\angle BCO = \angle DCO$,

$\therefore \angle OBC = \angle ODC$. I.

Then, $\because \angle ABC = \angle CDE$, Hy

and $\angle CDE =$ twice $\angle ODC$,

$\therefore \angle ABC =$ twice $\angle OBC$.

Hence OB bisects $\angle ABC$.

In the same way we can shew that OA, OE bisect the $\angle$s BAE, AED.

From O draw OF, OG, OH, OK, OL $\perp$ to AB, BC, C DE, EA.

Then, in △s GOC, HOC,

$\because \angle GCO = \angle HCO$, and $\angle OGC = \angle OHC$,

and OC is common,

$\therefore OG = OH$. I.

So also it may be shewn that OF, OL, OK are each $= OG$ or OH;

$\therefore OF$, OG, OH, OK, OL are all equal.

Hence a ⊙ described with centre O and radius OF will pass through G, H, K, L,

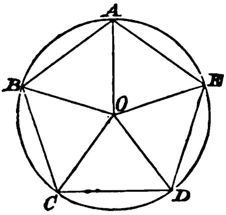
and will touch the sides of the pentagon,

$\because$ the $\angle$s at F, G, H, K, L are rt. $\angle$s.

Thus a ⊙ will be inscribed in the pentagon. Q.

PROPOSITION XIV. PROBLEM.

To describe a circle about a given regular pentagon.



Let $ABCDE$ be the given regular pentagon.

It is required to describe a ⊙ about the pentagon.

Bisect the $\angle$s BCD, CDE by the st. lines CO, DO, meeting in O.

Join OB, OA, OE.

Then it may be shewn, as in the preceding proposition, that

OB, OA, OE bisect the $\angle$s CBA, BAE, AED.

And $\because \angle BCD = \angle CDE$,

and $\angle OCD =$ half $\angle BCD$, and $\angle ODC =$ half $\angle CDE$,

$\therefore \angle OCD = \angle ODC$,

and $\therefore OD = OC$.

In the same way we may shew that OB, OA, OE each $= OD$ or OC;

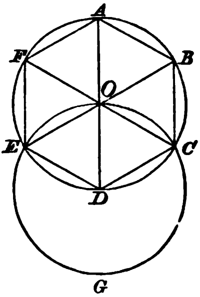
$\therefore OA$, OB, OC, OD, OE are all equal,

and a ⊙ described with centre O and radius OA will pass through B, C, D, E,

and will be described about the polygon. Q. E. F.

PROPOSITION XV. PROBLEM.

To inscribe a regular hexagon in a given circle.



Let $ABCDEF$ be the given ⊙, of which O is the centre.

It is required to inscribe a regular hexagon in the ⊙.

Draw the diameter AOD,

and with centre D and radius DO describe a ⊙ $EOCG$.

Join EO, CO, and produce them to B and F.

Join AB, BC, CD, DE, EF, FA.

Then ∵ O is the centre of ⊙ ACE, ∴ $OE = OD$;

and ∵ D is the centre of ⊙ GCE, ∴ $OD = DE$;

∴ OED is an equilateral △,

and ∴ ∠ EOD = the third part of two rt. ∠ s. I. 32.

So also ∠ DOC = the third part of two rt. ∠ s,

and ∴ ∠ BOC = the third part of two rt. ∠ s. I. 13.

Thus ∠ s EOD, DOC, BOC are all equal;

and to these the vertically opposite ∠ s BOA, AOF, FOE are equal; I. 15.

∴ ∠ s AOB, BOC, COD, DOE, EOF, FOA are all equal,

and ∴ arcs AB, BC, CD, DE, EF, FA are all equal; III. 26.

and ∴ chords AB, BC, CD, DE, EF, FA are all equal. III. 29.

Thus the hexagon $ABCDEF$ is equilateral.

Also ∵ each of its ∠ s = two-thirds of two rt. ∠ s,

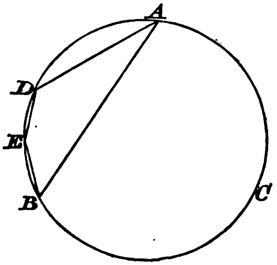
∴ the hexagon $ABCDEF$ is equiangular.

Thus a regular hexagon has been inscribed in the ⊙.

Q. E. F.

PROPOSITION XVI. PROBLEM.

To inscribe a regular quindecagon in a given circle.



Let *ABC* be the given ⊙.

It is required to inscribe in the ⊙ *a regular quindecagon.*

Let *AB* be the side of an equilateral △ inscribed in the ⊙, IV. 2.

and *AD* the side of a regular pentagon inscribed in the ⊙. IV. 11.

Then of such equal parts as the whole ○ce *ABC* contains fifteen,

arc *ADB* must contain three,

and arc *AD* must contain five,

and ∴ arc *DB*, their difference, must contain two.

Bisect arc *DB* in *E*. III. 30.

Then arcs *DE*, *EB* are each the fifteenth part of the whole ○ce.

If then chords, *DE*, *EB* be drawn,

and chords equal to them be placed all round the ○ce, IV. 1.

a regular quindecagon will be inscribed in the ⊙.

Q. E. F.

MISCELLANEOUS EXERCISES ON BOOK IV.

1. The perpendiculars let fall on the sides of an equilateral triangle from the centre of the circle described about it, are equal.

2. Inscribe a circle in a given regular octagon.

3. Shew that in the diagram of Prop. X. there is a second triangle which has each of two of its angles double of the third.

4. Describe a circle about a given rectangle.

5. Shew that the diameter of the circle which is described about an isosceles triangle, which has its vertical angle double of either of the angles at the base, is equal to the base of the triangle.

6. The side of the equilateral triangle, described about a circle, is double of the side of the equilateral triangle, inscribed in the circle.

7. A quadrilateral figure may have a circle described about it, if the rectangles contained by the segments of the diagonals be equal.

8. The square of the side on an equilateral triangle inscribed in a circle, is triple of the square on the side of the regular hexagon, inscribed in the same circle.

9. Inscribe a circle in a given rhombus.

10. *ABC* is an equilateral triangle inscribed in a circle; tangents to the circle at *A* and *B* meet in *M*. Shew that the diameter drawn from *M* bisects the angle *AMB*, and is itself trisected by the circumference.

11. Compare the areas of two regular hexagons, one inscribed in, the other described about, a given circle.

12. Inscribe a square in a given semicircle.

13. A circle being given, describe six other circles, each of them equal to it, and in contact with each other, and with the given circle.

14. Given the angles of a triangle, and the perpendiculars from any point on the three sides, construct the triangle.

15. Having given the radius of a circle, determine its centre, when the circle touches two given lines, which are not parallel.

16. If the distance between the centres of two circles, which cut one another at right angles, is equal to twice one of the radii, the common chord is the side of the regular hexagon, inscribed in one of the circles, and the side of the equilateral triangle, inscribed in the other.

17. Construct a square, having given the sum, or the difference, of the diagonal and the side.

18. If from O, the centre of the circle inscribed in a triangle ABC, OD, OE, OF be drawn perpendicular to the sides BC, CA, AB, respectively, and from any point P in OP, drawn parallel to AB, perpendiculars PQ, PR be drawn upon OD and OE respectively, or these produced, shew that the triangle QRO is equiangular to the triangle ABC.

APPENDIX.

QUESTIONS arising out of the Propositions, to which they are attached, have been proposed in the Euclid Papers to Candidates for Mathematical Honors since the year 1848.

A complete set of these questions, so far as they refer to Books I—IV, is here given. The figures preceding each question denote the particular Proposition to which the question was attached. It is expected that the solution of each question is to be obtained mainly by using the Proposition which precedes it, and that no Proposition which comes later in Euclid's order should be assumed.

Of some of the questions here given we have already made use in the preceding pages. As examples, however, of what has been hitherto expected of Candidates for Honors, and in order to keep the series of Papers complete, we have not hesitated to repeat them.

Euclid Papers set in the Mathematical Tripos.

1848. I. C. How does it appear that the two triangles are equiangular and equal to each other?

I. 34. If the two diagonals be drawn, shew that a parallelogram will be divided into four equal parts. In what case will the diagonal bisect the angle of a parallelogram?

III. 15. Shew that all equal straight lines in a circle will be touched by another circle.

III. 20. If two straight lines AEB, CED in a circle intersect in E, the angles subtended by AC and BD at the centre are together double of the angle AEC.

849. I. 1. By a method similar to that used in this problem, describe on a given finite straight line an isosceles triangle, the sides of which shall be each equal to twice the base.

II. 11. Shew that in Euclid's figure four other lines, beside the given line, are divided in the required manner.

IV. 4. Describe a circle touching one side of a triangle and the produced parts of the other two.

850. I. 34. If the opposite sides, or the opposite angles, of any quadrilateral figure be equal, or if its diagonals bisect each other, the quadrilateral is a parallelogram.

II. 14. Given a square, and one side of a rectangle which is equal to the square, find the other side.

III. 31. The greatest rectangle that can be inscribed in a circle is a square.

III. 34. Divide a circle into two segments such that the angle in one of them shall be five times the angle in the other.

IV. 10. Shew that the base of the triangle is equal to the side of a regular pentagon inscribed in the smaller circle of the figure.

851 I. 38. Let ABC, ABD be two equal triangles, upon the same base AB and on opposite sides of it: join CD, meeting AB in E: shew that CE is equal to ED.

I. 47. If ABC be a triangle, whose angle A is a right angle, and BE, CF be drawn bisecting the opposite sides respectively, shew that four times the sum of the squares on BE and CF is equal to five times the square on BC.

III. 22. If a polygon of an even number of sides be inscribed in a circle, the sum of the alternate angles together with two right angles is equal to as many right angles as the figure has sides.

1851. IV. 16. In a given circle inscribe a triangle, whose angles are as the numbers 2, 5 and 8.

1852. I. 42. Divide a triangle by two straight lines into three parts, which when properly arranged shall form a parallelogram whose angles are of given magnitude.

II. 12. Triangles are described on the same base and having the difference of the squares on the other sides constant: shew that the vertex of any triangle is in one or other of two fixed straight lines.

IV. 3. Two equilateral triangles are described about the same circle: shew that their intersections will form a hexagon equilateral, but not generally equiangular.

1853. I. 6. If lines be drawn through the extremities of the base of an isosceles triangle, making angles with it, on the side remote from the vertex, each equal to one-third of one of the equal angles, and meeting the sides produced, prove that three of the triangles thus formed are isosceles.

I. 29. Through two given points draw two lines, forming with a line, given in position, an equilateral triangle.

II. 11. In the figure, if H be the point of division of the given line AB, and DA be the side of the square which is bisected in E and produced to F, and if DH be produced to meet BF in L, prove that DL is perpendicular to BF, and is divided by BE similarly to the given line.

III. 32. Through a given point without a circle draw a chord such that the difference of the angles in the two segments, into which it divides the circle, may be equal to a given angle.

III. 36. From a given point as centre describe a circle cutting a given line in two points, so that the rectangle contained by their distances from a fixed point in the line may be equal to a given square.

1854. I. 43. If *K* be the common angular point of the parallelograms about the diameter, and *BD* the other diameter, the difference of the parallelograms is equal to twice the triangle *BKD*.

II. 11. Produce a given straight line to a point such that the rectangle contained by the whole line thus produced and the part produced shall be equal to the square on the given straight line.

III. 22. If the opposite sides of the quadrilateral be produced to meet in *P*, *Q*, and about the triangles so formed without the quadrilateral circles be described meeting again in *R*, shew that *P*, *R*, *Q* will be in one straight line.

IV. 10. Upon a given straight line, as base, describe an isosceles triangle having the third angle treble of each of the angles at the base.

1855. I. 20. Prove that the sum of the distances of any point from the three angles of a triangle is greater than half the perimeter of the triangle.

I. 47. If a line be drawn parallel to the hypotenuse of a right-angled triangle, and each of the acute angles be joined with the points where this line intersects the sides respectively opposite to them, the squares on the joining lines are together equal to the squares on the hypotenuse and on the line drawn parallel to it.

II. 9. Divide a given straight line into two parts, such that the square on one of them may be double of the square on the other, without employing the Sixth Book.

III. 27. If any number of triangles, upon the same base *BC*, and on the same side of it, have their vertical angles equal, and perpendiculars meeting in *D* be drawn from *B*, *C* upon the opposite sides, find the locus of *D*, and shew that all the lines which bisect the angle BDC pass through the same point.

1855. IV. 4. If the circle inscribed in a triangle *ABC* touch the sides *AB*, *AC* in the points *D*, *E*, and a straight line be drawn from *A* to the centre of the circle, meeting the circumference in *G*, shew that *G* is the centre of the circle inscribed in the triangle *ADE*.

1856. I. 34. Of all parallelograms, which can be formed with diameters of given length, the rhombus is the greatest.

II. 12. If *AB*, one of the equal sides of an isosceles triangle *ABC*, be produced beyond the base to *D*, so that $BD = AB$, shew that the square on *CD* is equal to the square on *AB* together with twice the square on *BC*.

IV. 15. Shew how to derive the hexagon from an equilateral triangle inscribed in the circle, and from this construction shew that the side of the hexagon equals the radius of the circle, and that the hexagon is double of the triangle.

1857. I. 35. *ABC* is an isosceles triangle, of which *A* is the vertex: *AB*, *AC* are bisected in *D* and *E* respectively; *BE*, *CD* intersect in *F*: shew that the triangle *ADE* is equal to three times the triangle *DEF*.

II. 13. The base of a triangle is given, and is bisected by the centre of a given circle, the circumference of which is the locus of the vertex: prove that the sum of the squares on the two sides of the triangle is invariable.

III. 22. Prove that the sum of the angles in the four segments of the circle, exterior to the quadrilateral, is equal to six right angles.

IV. 4. Circles are inscribed in the two triangles formed by drawing a perpendicular from an angle of a triangle upon the opposite side, and analogous circles are described in relation to the two other like perpendiculars: prove that the sum of the diameters of the six circles together with the sum of the sides of the original triangle is equal to twice the sum of the three perpendiculars.

1858. I. 28. Assuming as an axiom that two straight lines cannot both be parallel to the same straight line, deduce Euclid's sixth postulate as a corollary of the proposition referred to.

II. 7. Produce a given straight line, so that the sum of the squares on the given line and the part produced may be equal to twice the rectangle contained by the whole line thus produced and the produced part.

III. 19, Describe a circle, which shall touch a given straight line at a given point and bisect the circumference of a given circle.

1859. I. 41. Trisect a parallelogram by straight lines drawn from one of its angular points.

II. 13. Prove that, in any quadrilateral, the squares on the diagonals are together equal to four times the sum of the squares on the straight lines joining the middle points of opposite sides.

III. 31. Two equal circles touch each other externally, and through the point of contact chords are drawn, one to each circle, at right angles to each other: prove that the straight line, joining the other extremities of these chords, is equal and parallel to the straight line joining the centres of the circles.

IV. 4. Triangles are constructed on the same base, with equal vertical angles: prove that the locus of the centres of the escribed circles, each of which touches one of the sides externally and the other side and base produced, is an arc of a circle, the centre of which is on the circumference of the circle circumscribing the triangles.

1860. I. 35. If a straight line *DME* be drawn through the middle point *M* of the base *BC* of a triangle *ABC*, so as to cut off equal parts *AD*, *AE* from the sides *AB*, *AC*, produced if necessary, respectively, then shall *BD* be equal to *CE*.

II. 14. Shew how to construct a rectangle which shall be equal to a given square; (1) when the sum, and (2) when the difference of two adjacent sides is given.

III. 36. If two chords *AB*, *AC* be drawn from any point *A* of a circle, and be produced to *D* and *E*, so that the rectangle *AC*, *AE* is equal to the rectangle *AB*, *AD*, then, if *O* be the centre of the circle, *AO* is perpendicular to *DE*.

IV. 10. If *A* be the vertex, and *BD* the base of the constructed triangle, *D* being one of the points of intersection of the two circles employed in the construction, and *E* the other, and *AE* be drawn meeting *BD* produced in *F*, prove that *FAB* is another isosceles triangle of the same kind.

1861. I. 32. If *ABC* be a triangle, in which *C* is a right angle, shew how, by means of Book I, to draw a straight line parallel to a given straight line so as to be terminated by *CA* and *CB* and bisected by *AB*.

II. 13. If *ABC* be a triangle, in which *C* is a right angle, and *DE* be drawn from a point *D* in *AC* at right angles to *AB*, prove, without using Book III, that the rectangles *AB*, *AE* and *AC*, *AD* will be equal.

III. 32. Two circles intersect in *A* and *B*, and *CBD* is drawn perpendicular to *AB* to meet the circles in *C* and *D*; if *EAF* bisect either the interior or exterior angle between *CA* and *DA*, prove that the tangents to the circles at *E* and *F* intersect in a point on AB produced.

1861. IV. 4. Describe a circle touching the side BC of the triangle ABC, and the other two sides produced, and prove that the distance between the points of contact of the side BC with the inscribed circle, and the latter circle, is equal to the difference between the sides AB and AC.

1862. I. 4. Upon the sides AB, BC, and CD of a parallelogram $ABCD$, three equilateral triangles are described, that on BC towards the same parts as the parallelogram, and those on AB, CD, towards the opposite parts. Prove that the distances of the vertices of the triangles on AB, CD, from that on BC, are respectively equal to the two diagonals of the parallelogram.

III. 10. Divide a given straight line into two parts, so that the squares on the whole line and on one of the parts may be together double of the square on the other part.

III. 28. A triangle is turned about its vertex, until one of the sides intersecting in that vertex is in the same straight line as the other previously was: prove that the line, joining the vertex with the point of intersection of the two positions of the base produced if necessary, bisects the angle between these two positions.

IV. 10. Prove that the smaller of the two circles, employed in Euclid's construction, is equal to the circle described about the required triangle.

863. I. 47. Two triangles ABC, $A'B'C'$ have their sides respectively parallel. BB_1, CC_1 are drawn perpendicular to $B'C'$; CC_2, AA_2 to CA'; and AA_3, BB_3 to $A'B'$. Prove that the sum of the squares on AB_1, BC_2, CA_3 together, is equal to the sum of those on AC_1, BA_2, CB_3 together.

II. 11. Divide a given straight line into two parts, such that the rectangle contained by the whole and one part may be equal to that contained by the other part and a given straight line.

1863. III. 28. Two equal circles intersect in A, B; PQT perpendicular to AB meets it in T, and the circles in P, Q. AP, BQ meet in R; AQ, BP in S: prove that the angle RTS is bisected by TP.

1864. I. 38. If a quadrilateral figure have two sides parallel, and the parallel sides be bisected, the line joining the points of bisection shall pass through the point in which the diagonals cut one another.

II. 14. Divide a given straight line (when possible) into three parts such that the rectangle contained by two of them shall be equal to a given rectilineal figure, and that the squares on these two parts shall together be equal to the square on the third.

III. 36. If from a given point A without a given circle any two straight lines APQ, ARS, be drawn, making equal angles with the diameter which passes through A, and cutting the circle in P, Q, and R, S, respectively, then PS, QR, shall cut one another in a given point.

IV. 11. If a figure of any odd number of sides have all its angular points on the same circle, and all its angles equal, then shall its sides be equal.

1865. I. 20. Give a geometrical construction for finding a point in a given straight line, the difference of the distances of which from two given points on the same side of the line shall be the greatest possible.

II. 12. The base BC of an isosceles triangle ABC is produced to a point D; AD is joined, and in AD a point E is taken, such that the rectangle AD, AE, is equal to the square on either of the equal sides AB, AC, of the triangle:

prove that the rectangle *BD*, *CD* is equal to the rectangle *AD*, *ED*.

865. III. 18. A given straight line is drawn at right angles to the straight line joining the centres of two given circles: prove that the difference between the squares on two tangents drawn, one to each circle, from any point on the given straight line, is constant.

IV. 5. Having given one side of a triangle, and the centre of the circumscribed circle, determine the locus of the centre of the inscribed circle.

866. I. 33. Prove that a quadrilateral, which has two opposite sides and two opposite obtuse angles equal, is a parallelogram.

Shew that the figure is not necessarily a parallelogram, if the equal angles are acute.

II. 9. Prove this also by superposition of the squares or their halves.

III. 22. If four circles be drawn, each passing through three out of four given points, the angle between the tangents at the intersection of two of the circles is equal to the angle between the tangents at the intersection of the other two circles.

IV. 2. In a given circle inscribe a triangle such that two of the sides of the triangle shall pass through given points and the third side be at a given distance from the centre of the given circle.

1867. I. 16. Any two exterior angles of a triangle are together greater than two right angles.

I. 43. What is the greatest value which these complements, for a given parallelogram, can have?

II. 11. Divide a given straight line into two parts such that the squares on the whole line and on one of the parts shall be together double of the square on the other part.

14—2

1867. III. 22. If the chords, which bisect two angles of a triangle inscribed in a circle, be equal, prove that either the angles are equal, or the third angle is equal to the angle of an equilateral triangle.

1868. I. 41. $OKBM$ and $OLDN$ are parallelograms about the diameter of a parallelogram $ABCD$. In MN, which is parallel to BA, take any point P and prove that, if PC, produced if necessary, meet KL in Q, BP will be parallel to DQ.

II. 12. In a triangle ABC, D, E, F are the middle points of the sides BC, CA, AB respectively, and K, L, M are the feet of the perpendiculars on the same sides from the opposite angles. Prove that the greatest of the rectangles contained by BC and DK, CA and EL, AB and FM is equal to the sum of the other two.

III. 35. Through a point, within a circle, draw a chord, such that the rectangle contained by the whole chord and one part may be equal to a given square.

Determine the necessary limits to the magnitude of this square.

IV. 4. If two triangles ABC, $A'B'C'$ be inscribed in the same circle, so that AA', BB', CC' meet in one point O, prove that, if O be the centre of the inscribed circle of one of the triangles, it will be the centre of the perpendiculars of the other.

1869. I. 40. ABC is a triangle, E and F are two points; if the sum of the triangles ABE and BCE be equal to the sum of the triangles ABF and BCF, then under certain conditions EF will be parallel to AC. Find these conditions, and determine when the difference instead of the sum of the triangles must be taken.

1869. II. 11. Shew that the point of section lies between the extremities of the line.

III. 33. An acute-angled triangle is inscribed in a circle, and the paper is folded along each of the sides of the triangle: shew that the circumferences of the three segments will pass through the same point. State the equivalent proposition for an obtuse-angled triangle.

IV. 11. Shew that the circles, each of which touches two sides at the extremities of a third, meet in a point.

1870. I. 26. $ABCD$ is a square and E a point in BC; a straight line EF is drawn at right angles to AE, and meets the straight line, which bisects the angle between CD and BC produced, in a point F: prove that AE is equal to EF.

II. 9. The diagonals of a quadrilateral meet in E, and F is the middle point of the straight line joining the middle points of the diagonals: prove that the sum of the squares on the straight lines joining E to the angular points of the quadrilateral is greater than the sum of the squares on the straight lines joining F to the same points by four times the square on EF.

III. 32. AB, CD are parallel diameters of two circles, and AC cuts the circles in P, Q: prove that the tangents to the circles at P, Q are parallel.

IV. 10. Hence shew how to describe an equilateral and equiangular pentagon about a circle without first inscribing one.

1871. I. 38. Through the angular points A, B, C, of a triangle are drawn three parallel straight lines meeting the opposite sides in A', B', C' respectively: prove that the triangles $AB'C'$, $BC'A'$, $CA'B'$ are all equal.

II. 10. Produce a given straight line so that the square on the whole line thus produced may be double the square on the part produced.

1871. III. 32. The opposite sides of a quadrilateral inscribed in a circle are produced to meet in P, Q, and about the four triangles thus formed circles are described: prove that the tangents to these circles at P and Q form a quadrilateral equal in all respects to the original, and that the line joining the centres of the circles, about the two quadrilaterals, bisects PQ.

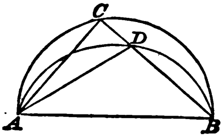
IV. 5. A triangle is inscribed in a given circle so as to have its centre of perpendiculars at a given point: prove that the middle points of its sides lie on a fixed circle.

We here insert Euclid's proofs of Props. 23, 24 of Book III. first observing that he gives the following definition of similar segments:—

Def. Similar segments of circles are those in which the angles are equal, or which contain equal angles.

Book III. Proposition XXIII. Theorem.

Upon the same straight line, and upon the same side of it, there cannot be two similar segments of circles, not coinciding with each other.



If it be possible, on the same base AB, and on the same side of it, let there be two similar segments of ⊙s, ABC, ABD, which do not coincide.

Because ⊙ ADB cuts ⊙ ACB in pts. A and B, they cannot cut one another in any other pt., and ∴ one of the segments must fall within the other.

Let ADB fall within ACB.

Draw the st. line BDC and join CA, DA.

Then ∵ segment ADB is similar to segment ACB,

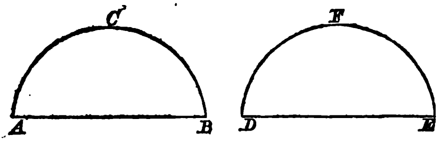
∴ $\angle ADB = \angle ACB$.

Or the extr. ∠ of a △ = the intr. and opposite ∠, which is impossible;

∴ the segments cannot but coincide. Q. E. D.

BOOK III. PROPOSITION XXIV. THEOREM.

Similar segments of circles, upon equal straight lines, are equal to one another.



Let *ABC*, *DEF* be similar segment of ⊙s on equal st. lines *AB*, *DE*.

Then must segment ABC = *segment* DEF.

For if segment *ABC* be applied to segment *DEF*, so that *A* may be on *C* and *AB* on *DE*, then *B* will coincide with *E*, and *AB* with *DE*;

∴ segment *ABC* must also coincide with segment *DEF*; III. 23.

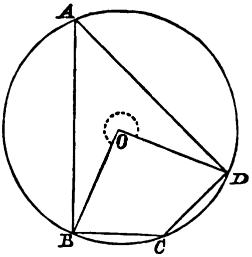
∴ segment *ABC* = segment *DEF*. Ax. 8.

Q. E. D.

We gave one Proposition, *C*, page 150, as an example of the way in which the conceptions of Flat and Reflex Angles may be employed to extend and simplify Euclid's proofs. We here give the proofs, based on the same conceptions, of the important propositions XXII. and XXXI.

PROPOSITION XXII. THEOREM.

The opposite angles of any quadrilateral figure, inscribed in a circle, are together equal to two right angles.



Let *ABCD* be a quadrilateral fig. inscribed in a ⊙.

Then must each pair of its opposite ∠s be together equal to two rt. ∠s.

From *O*, the centre, draw *OB*, *OD*.

Then ∵ ∠*BOD* = twice ∠*BAD*, III. 20.

and the reflex ∠*DOB* = twice ∠*BCD*, III. C. p. 150.

∴ sum of ∠s at *O* = twice sum of ∠s *BAD*, *BCD*.

But sum of ∠s at *O* = 4 right ∠s; I. 15, Cor. 2.

∴ twice sum of ∠s *BAD*, *BCD* = 4 right ∠s;

∴ sum of ∠s *BAD*, *BCD* = two right ∠s.

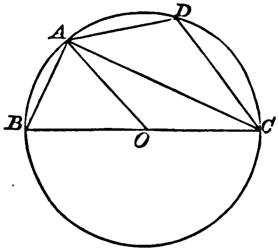
Similarly, it may be shewn that

sum of ∠s *ABC*, *ADC* = two right ∠s.

Q.E.D.

PROPOSITION XXXI. THEOREM.

In a circle, the angle in a semicircle is a right a and the angle in a segment greater than a semicircle i than a right angle; and the angle in a segment less t semicircle is greater than a right angle.



Let ABC be a ⊙, of which O is the centre and diameter.

Draw AC, dividing the ⊙ into the segments ABC, A

Join BA, AD, DC.

Then must the ∠ *in the semicircle* BAC *be a rt.* ∠ ∠ *in segment* ABC, *greater than a semicircle, less t rt.* ∠, *and* ∠ *in segment* ADC, *less than a semi greater than a rt.* ∠.

First, ∵ the flat angle BOC = twice ∠ BAC, III. C. p

∴ ∠ BAC is a rt. ∠.

Next, ∵ ∠ BAC is a rt. ∠,

∴ ∠ ABC is less than a rt. ∠.

Lastly, ∵ sum of ∠s ABC, ADC = two rt. ∠s, I

and ∠ ABC is less than a rt. ∠,

∴ ∠ ADC is greater than a rt. ∠.